Summary and Clarifications about Conjugates and Classes

Conjugacy

Any two elements $\boxed{g_i, g_j \in G(A,\circ) \text{ are cojugates if, for an element } h_{ij} \in G,\ g_i = h_{ij} g_j h_{ij}^{-1}}$

↳ Say $H \leq G$. H is conjugacy closed if, for any two elements $g_i, g_j \in H$, $g_i = h_{ij} g_j h_{ij}^{-1}$ with $h_{ij} \in H$

i.e. Any subgroup is conjugacy closed if any two elements of the subgroup that are conjugate in the group are also conjugate in the subgroup

↳ Conjugacy is an equivalence relation ($\sim$) which must satisfy:

1) Reflexivity i.e. $a \sim a$, $a \in S$
2) Symmetry i.e. if $a \sim b$, then $b \sim a$ for $a, b \in S$
3) Transitivity i.e. if $a \sim c$ and $b \sim c$, then $a \sim b$ for $a, b, c \in S$

Classes

A class K of a group $G(G,\circ)$ is a subset of G in which all elements of K are conjugate to each other and and no elements of $G \setminus K$ are conjugates to any elements of K. That is, conjugacy leads to the partition of G into disjoint classes given by the set of classes $\{K_i\}$ i.e. every element is in at least one class

- If $\forall a, b \in K$, $b = hah^{-1}$ where $h \in G$
- If $\forall a \in K$, $\nexists c \in G \setminus K \mid c = hah^{-1}$ or $a = hch^{-1}$ i.e. c is not related to a by conj. if $a \in K$ and $c \in G \setminus K$

Class Definition: $\boxed{\underset{K_a}{(a)} = \{ b \mid b = hah^{-1},\ h \in G \text{ and } a, b \in (a)\}}$

Class Properties:

Group G with set of classes $\{K_i\}$

1. For $\forall g \in G$, $g \in K_i$ and only K_i i.e. Classes either completely overlap or are completely disjoint
2. $\{e\}$ forms a (single-element) class.
3. If G is abelian, all classes correspond of single elements

Forbidde by Definition ↑

① i.e. g in one class and no more than one $\Longrightarrow$ If g_2 where to be in K_1 and K_2, conj. relation between K_1 and $G \setminus K_1$

② i.e. e its is own and only conj. $\Longrightarrow$ $g = heh^{-1} = e\,hh^{-1} = e \quad \forall h \in G$

③ i.e. g its is own and only conj. if $g_i \circ g_j = g_j \circ g_i \quad \forall g_i, g_j \in G$ $\Longrightarrow$ $g_i = h g_j h^{-1} = g_j h h^{-1} = g_j \quad \forall g_j, h \in G$ if G Abelian

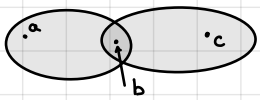
Why these properties?

Is $a \in (a)$? yes, as $b \in (a) \mid b = gag^{-1}$, $g \in G$ and if $g = e$, $b = a$

If $b \in (a)$ is $a \in (b)$? yes as $b = gag^{-1}$ implies $a = g^{-1}bg = g' b (g')^{-1}$

If $a \in (b) \wedge b \in (c)$, $a \in (c)$? yes. If we exploit that if $a \in (b)$ the $b \in (a)$ we have: $b = gag^{-1} = hch^{-1}$ or $a = g^{-1}hch^{-1}g = g'c(g')^{-1}$

As $a \in (b)$ we have $b \in (a)$ $\Longrightarrow$ $a, b \in (a) \cap (b)$

This implies $a, b, c \in (a) \cap (b) \cap (c)$ $\Longrightarrow$ This forbids the relation $a \sim b$, $b \sim c$ with $a \nsim c$ i.e.  is impossible

As a result $(a) = (b) = (c)$ $\Longrightarrow$ Classes can only overlap completely or be completely disjoint

Center

A center of a group $G(G,\circ)$ is the subset of G that commute with all other elements of g

i.e. $\boxed{Z(G) = \{ z \in G \mid zg = gz \ \forall g \in G\}}$

Properties:

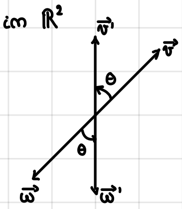
- Abelian
- Closed under conjugation and all of its elements form a class by themselves

↳ If $z \in Z(G)$ the conjugate $b = hzh^{-1} = z\,hh^{-1} = z \quad \forall h \in G$

Geometrical Interpretation of Conjugacy

Consider two vectors $\vec v, \vec v\,', \vec w, \vec w\,' \in \mathbb{R}^n$ and the group $G = (\{c,h,\dots\},\circ)$

e.g. in $\mathbb{R}^2$



$c \equiv$ rotation by θ

$h \equiv$ space inversion

Assume now that $\vec v\,' = c\vec v$, $\vec w = h\vec v$ and $\vec w\,' = h\vec v\,'$

Then: $\vec v\,'\cdot\vec v = \sum_i\sum_j v'_i v_j \delta^i_j = |\vec v\,'||\vec v|\cos\theta_1$, $\vec w\,'\cdot\vec w = |\vec w\,'||\vec w|\cos\theta_2$

$\vec w\,' = h\vec v\,' = hc(h^{-1}h)\vec v = hch^{-1}\vec w$

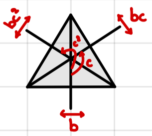
The conjugate hch^{-1} is the operation that maps $\vec w \longmapsto \vec w\,'$

where c maps $\vec v \longmapsto \vec v\,'$ and h maps $\vec v \longmapsto \vec w$, $\vec v\,' \longmapsto \vec w\,'$

If we require the length of and angle between $\vec v\,', \vec v$ to be equal to the length of and angle between $\vec w\,', \vec w$ we have that c and hch^{-1} are orthonormal transformations and $\vec v\,'\cdot\vec v = \vec w\,'\cdot\vec w = |v|^2\cos\theta \Longrightarrow$ This is valid for rotations and reflections

See Isometries for more on orthonormal transformations

Example: Equilateral triangle



Symmetry group: $D_3 = \{e,c,c^2,b,bc,bc^2\}$, $b^2=c^3=(bc)^2=e$

↳ $c^{-1}=c^2$ $\quad b^{-1}=b$

↳ $cb = b^{-1}ec^{-1} = bc^2$ and $c^2b = c(bc) = (cb)c^2 = bc$

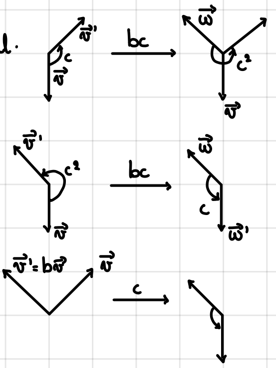
Brute force:

↳ (c):

1. $bcb^{-1} = b^2c^2 = c^2$
2. $c^2c(c^2)^{-1} = c^3c = c$
3. $(bc)c(bc)^{-1} = (bc)c(bc) = bcbc^3 = b^2c^2 = c^2$
4. $(bc^2)c(bc^2)^{-1} = bc^2cbc^2 = bc^2bc^4 = b(c^2b)c = b^2c^2 = c^2$
5. $cc(c^{-1}) = c$

Similarly: $(e)=\{e\}$, $(c)=\{c,c^2\}$, $(b)=\{b,bc,bc^2\}$

Geometrical.

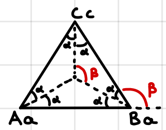


i.e. $c^2: \vec w \longmapsto \vec w\,'$ so $c^2 = (bc)c(bc)^{-1}$

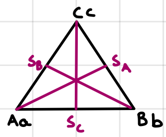
i.e. $c: \vec w \longmapsto \vec w\,'$ so $c = (bc)c^2(bc)^{-1}$

Examples

Example 10



$\alpha = 30°$ $\beta = 120°$



A, B, C : Vertices of triangle

a, b, c : Points corresponding to A, B, C in the not-yet transformed reference frame

Rotations:

Equilateral triangle is symmetric under rotation $R^m = m\beta$, $m \in \mathbb{Z}$.

As $R^m = R^{3k+m}$ $\forall k \in \mathbb{Z}$ we only have three distinct rotations

$$\left.\begin{array}{l} \cdot\ E = R^0 = R^3 = \dots = e^{i0} = 1 \\ \cdot\ R = R^4 = R^7 = \dots = e^{i(2/3)\pi} \\ \cdot\ R^2 = R^5 = R^8 = \dots = e^{i(4/3)\pi} \end{array}\right\} R$$

These rotations form the group $C_3(R, \times)$

- Closure: $\forall R^m, R^p \in R$ we have $R^m + R^p = R^{m+p} = e^{i(m+p)(\pi/3)} = e^{i2\pi(m+p)/6} = R^{(m+p)} = R^{(m+p)\bmod(6)} = R^{m+3k}$
- Associative: $\forall R^m, R^p, R^d \in R$ we have $R^d + (R^m + R^p) = R^{d+m+p} = (R^d + R^m) + R^p$
- Neutral : E
- Inverse : $\forall R^m \in R, \exists (R^m)^{-1} \in R \mid R^m \times (R^m)^{-1} = E$ i.e. $(R^m)^{-1} = R^{3-m}$

Reflections

Equilateral triangle is symmetric under reflections about any bisectrix given by S_A, S_B and S_c

These then form three different sets in which:

- S_α : Reflection $\quad S_\alpha = \{E, S_\alpha\}$, $\alpha = \{A, B, C\}$
- S_α^2 : Neutral element

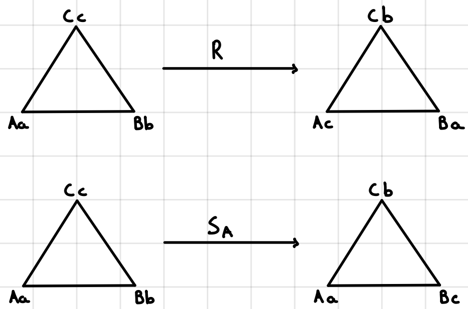
Groups: $S_\alpha(S_\alpha, \times)$

Total group D_3

$D_3 = C_3 \times S_A \times S_B \times S_c$

×	E	R	R^2	S_A	S_B	S_c
E	E	R	R^2	S_A	S_B	S_c
R	R	R^2	E	S_c	S_A	S_B
R^2	R^2	E	R	S_B	S_c	S_A
S_A	S_A	S_B	S_c	E	R^2	R
S_B	S_B	S_c	S_A	R	E	R^2
S_c	S_c	S_A	S_B	R^2	R	E

D_3 not Abelian



i.e. $S_A R$: (Aa, Bb, Cc) $\longmapsto$ (Ac, Bb, Ca) or S_B

$S_A R^2$: (Aa, Bb, Cc) $\longmapsto$ (Ab, Ba, Cc) or S_c

$R S_A$: (Aa, Bb, Cc) $\longmapsto$ (Ab, Ba, Cc) or S_c

$S_A S_B$: (Aa, Bb, Cc) $\longmapsto$ (Ac, Ba, Cb) i.e. R

$S_B S_A$: (Aa, Bb, Cc) $\longmapsto$ (Ab, Bc, Ca) i.e. R^2

$S_B S_c$: (Aa, Bb, Cc) $\longmapsto$ (Ac, Ba, Cb) i.e. R

$S_c S_B$: (Aa, Bb, Cc) $\longmapsto$ (Ab, Bc, Ca) i.e. R^2

Proper Subgroups: $R = \{E, R, R^2\}$ $\quad S_A = \{E, S_A\}$ $\quad S_B = \{E, S_B\}$ $\quad S_c = \{E, S_c\}$

These subgroups are:

- Abelian
- Proper because they are not the same as D_3 nor do they only contain $\{e\}$

$S_B S_A$: (Aa, Bb, Cc) $\xmapsto{S_A}$ (Aa, Bc, Cb) $\xmapsto{S_B}$ (Ab, Bc, Ca) i.e. R^2

Classes of D^3

$R^{-1} = R^2 \quad (R^2)^{-1} = R \quad S_\alpha^{-1} = S_\alpha$

Use given Latin square

$R^p = h R^m h^{-1}, \; h \in D_3 \xRightarrow[\text{Square}]{\text{From Latin}}$ $h = R^m \quad h^{-1} = R^{3-m}$ then $R^p = R^{3+m} = R^m \quad p = m$ that is R and R^2 are their own conj.

$h = S_\alpha \quad h^{-1} = S_\alpha$ then $R^p = S_\alpha R S_\alpha = S_\alpha S_{\alpha+1} = R^2$ i.e. $p = 2$ e.g. $R^2 = S_A R S_A = S_A S_B$

$R^p = S_\alpha R^2 S_\alpha = S_\alpha S_{\alpha+2} = R$ i.e. $p = 1$ e.g. $R = S_A R^2 S_A = S_A S_C$

However R^p are not related to S_α by conj as $h S_\alpha h^{-1}$ is always an S_α element according to the latin square

Classes: $(E) = \{E\}$, $(R) = \{R, R^2\}$ and $(S) = \{S_A, S_B, S_C\}$

Isometries

Isometry

Definition: A transformation T is isometric if it maintains the distance between two points invariant

e.g. $\vec{x}, \vec{y} \in \mathbb{R}^3$, $d = |T\vec{x} - T\vec{y}| = |\vec{x} - \vec{y}|$

Definition: The set of all isometries (i.e. isometric transformations) of the vector space $\mathbb{R}^3$ is known as the Euclidean Group $E(3)$ or $ISO(3)$

There are two main subgroups of $E(3)$: $O(3)$ and T

O(3) Group

$O(m)$ is the group of $m \times m$ orthogonal matrices with matrix multiplication as its composition law.

An orthogonal matrix is a real matrix Q that satisfies $QQ^T = I$ i.e. $Q^T = Q^{-1}$ $\Longrightarrow$ $\det(I) = \det(Q)\det(Q^T) = \det(Q)^2 = 1$ and $\det(Q) = \pm 1$

Therefore: $O(m) = \{ Q \in GL(m, \mathbb{R}^m) \mid QQ^T = Q^T Q = I \}$ where $GL(m, \mathbb{R}^m)$ is the General "Euclidean" Linear Group of all linear, invertible $m \times m$ matrix transformations

If $\det(Q) = 1$, $Q \in SO(m)$ where $SO(m) < O(m)$

Transformations in $O(m)$ maintain length and origin invariant $\Longrightarrow$ Point Groups are finite subgroups of the continuous group $O(m)$

Basis Transformation

For $\forall \vec{x} \in \mathbb{R}^m$, $\vec{x} = \sum_{i=1}^{m} x_i \hat{e}_i$ where $\hat{e}_i$ is a vector in an orthonormal basis $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_m\}$

Applying Linear Transformation R: $R: \hat{e}_i \longmapsto \hat{e}'_j = \sum R_{ij} \hat{e}_i$

As a result: $\vec{x} = \sum_i x_i \hat{e}_i = \sum x'_j \hat{e}'_j = \sum_i \sum_j x'_j R_{ij} \hat{e}_i$ i.e. $x_i = \sum_j x'_j R_{ij}$ or $\vec{x}_{old} = R \vec{x}_{new}$

Isometry:

For transformation to be isometric: $|\vec{x}_{old}| = |\vec{x}_{new}|$ i.e. $\sum_i x_i^2 = \sum_j x'^2_j$

As $\sum_i x_i^2 = \sum_i \sum_j x'_j R_{ij} \sum_k x'_k R_{ik} = \sum_i \sum_j \sum_k (x'_j x'_k) R_{ij} R_{ik} = \sum_i \sum_j \sum_k (x'_j x'_k)(R^T_{ji} R_{ik})$ and $\sum x_i^2 = \sum x'^2_j$ if $R^T_{ji} R_{ik} = \delta_{jk}$ or $R^T R = I$

That is, $R \in O(m)$

As two orthogonal transf result in an orthogonal transformation as $R_1^{-1} R_2^{-1} = R_1^T R_2^T = (R_1 R_2)^T = (R_1 R_2)^{-1}$, $O(m)$ forms a group

Space Inversion

Space inversion reverses direction of basis vectors i.e. $P\hat{e}_i = -\hat{e}_i$ and thus $P^2 = I$ and $P = P^{-1}$

Elements of O(3)

$\det(R)$, $R \in O(m)$ $\to$ $+1$, rotations; -1, translations

Let $A \in O(3)$ with $\det(A) = -1$. Then $R = AP \in SO(3)$ and $A = RP \in O(3)$

Any element of $O(3)$ can be written as a rotation $R \in SO(3)$ or a space inversion followed by a rotation i.e. $RP \in O(3)$

Rotations R form the subgroup $SO(3)$ as they satisfy all groups properties. On the other hand, reflections do not form a subgroup as two reflections leave the system invariant i.e. Applying a reflection twice would shoot out of the set and thus not satisfying closure

In order to describe the symmetry group of an unoriented circle we need 2 circles: One for rotations and one for the results of reflections

Conjugates

Say $R(\alpha), PR(\alpha) \in O(m)$

- R is a rotation in $\mathbb{R}^m$ i.e. $\det(R) = +1$
- PR is a reflection i.e. $\det(PR) = -1$

Reflections and Rotations can only be conjugate to other rotations and reflections respectively

Say: $PR_2 = hR_1h^{-1}$, $h \in O(m)$ i.e. $\det(PR_2) = \det(h)^2 \det(R_1)$

As $\det(h) = \pm 1$, we have $\det(R_1) = \det(PR_2)$ but we know this is not true

However: $R_2 = hR_1h^{-1}$, $PR_2 = hPR_1h^{-1}$ are perfectly fine for $\forall h \in O(m)$ i.e. rotation and reflection classes need not be conjugacy closed

Translations

Translations $T_{\vec{a}}: \vec{x} \longmapsto \vec{x} + \vec{a}$ are not linear transformations

While $T_{\vec{a}}$ maps $V \longmapsto W$, it does not preserve addition and scalar multiplication as:

- $f(\vec{u}+\vec{v}) \neq f(\vec{u}) + f(\vec{v})$ e.g. $\vec{x}+\vec{y}+\vec{a} \neq (\vec{x}+\vec{a}) + (\vec{y}+\vec{a})$
- $f(c\vec{u}) \neq cf(\vec{u})$ e.g. $c\vec{x}+\vec{a} \neq c\vec{x} + c\vec{a}$

Nonetheless, they maintain length invariant

$T: \vec{0} \longrightarrow \vec{0} + \vec{a}$ (Origin changes)

$T: T\vec{x} - T\vec{y} = \vec{x} - \vec{y}$

T is abelian as $T_{\vec{a}}(T_{\vec{b}}\vec{x}) = T_{\vec{a}}(\vec{x}+\vec{b}) = \vec{x}+\vec{a}+\vec{b} = T_{\vec{b}}(T_{\vec{a}}\vec{x})$

T is generally of infinite order given the infinite choice of translational vectors

Every $T \in E(3)$ can be uniquely written as a rotation/reflection followed by a translation i.e. $T = T_{\vec{a}} O = (O, \vec{a})$, $\vec{a} \in \mathbb{R}^3$, $O \in O(3)$

Homo and Isomorphisms

Important Definitions

- Injective i.e. 1-to-1 : A function is said to be injective if it maps distinct elements of its domain to distinct elements of its image
- Surjective i.e. onto : A surjective function f maps at least one element of its domain X to one element of its codomain Y s.t. $Y = \mathrm{Im}(f)$
  i.e. If $f: X \longrightarrow Y$, f is surjective if $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$ (Surjection can always be achieved by restricting Y to $\mathrm{Im}(f)$)
  If not surjective it is said to be "into"
- Bijective : A function f is said to be bijective if it is injective and surjective such that to every element in its domain there corresponds one and only one element in its codomain Y and viceversa. A function can thus be bijective if and only if it is invertible

Mapping a group to other group(s)

Some groups $G = (G', \circ)$ can be mapped to another group $G' = (G', \cdot)$ by means of a function ϕ i.e. $\phi: G \longmapsto G'$

↳ The mapping is said to be "homomorphic" if $\boxed{\forall g_1, g_2 \in G: \ \phi(g_1 \circ g_2) = \phi(g_1) \cdot \phi(g_2)}$

- Homomorphisms tend to be many-to-one and not always onto
- If the homorphic mapping ϕ is bijective (i.e. 1-to-1) it is said to be "isomorphic" i.e. $G \cong G'$

Kernel Ker(ϕ)

If e' is the identity in G' and G' is homomorphic to G, the kernel (ker) K is given by: $\boxed{K = \{\forall g \in G \mid \phi(g) = e'\}}$

In order to have a homorphism, ker(ϕ) must be made by complete classes of G

If $\ker\phi \neq \{e\}$, ϕ is not injective and thus not an isomorphism:

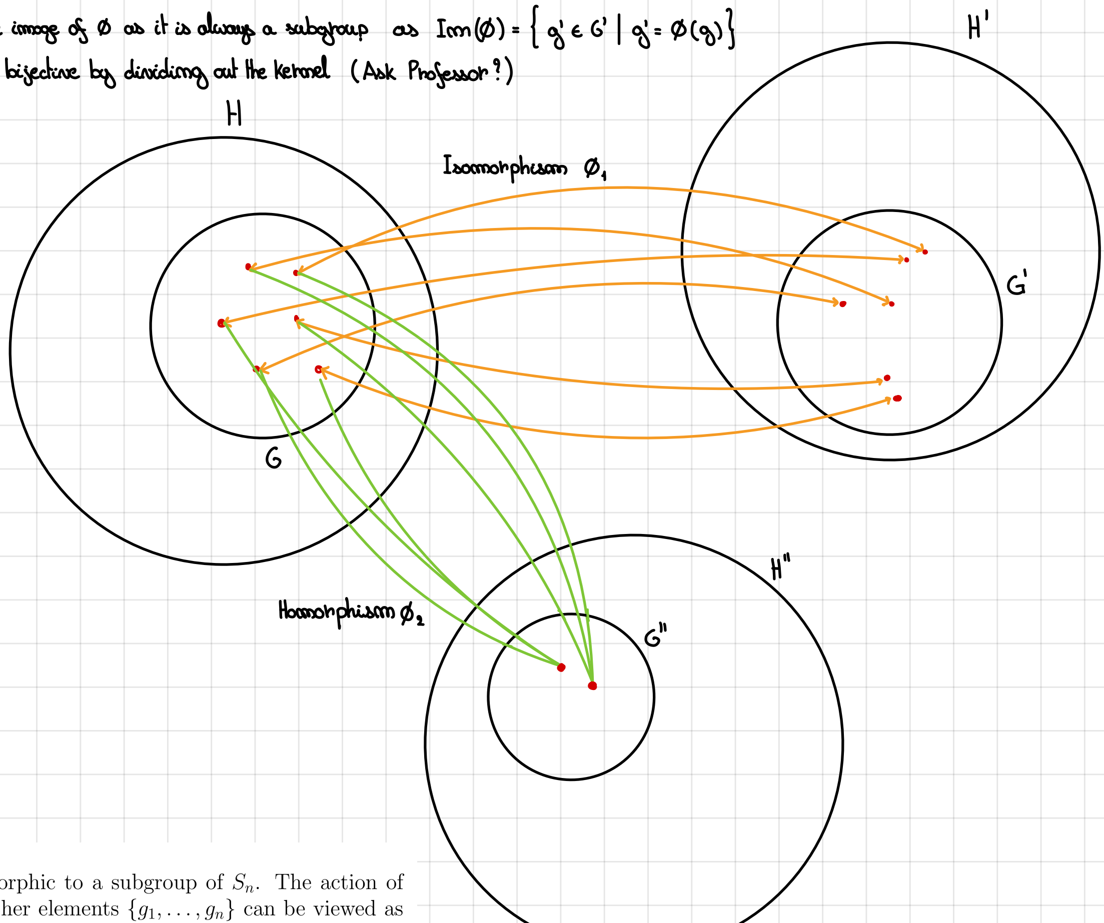
Proof: For $\phi: g \longrightarrow g'$ where $g \in G$, $g' \in G'$

It follows that if $g_1, g_2 \longrightarrow g'$, $g_1 g_2^{-1} \longrightarrow e'$ s.t. $g_1 g_2^{-1} \in \ker(\phi)$

If $g_1 \neq g_2$, ϕ is not injective and $g_1 g_2^{-1} \neq e$ s.t. $\ker\phi \neq \{e\}$

N.B. One can always restrict G' to the image of ϕ as it is always a subgroup as $\mathrm{Im}(\phi) = \{ g' \in G' \mid g' = \phi(g)\}$

N.B.2. One can make any mapping bijective by dividing out the kernel (Ask Professor?)



Cayley's Theorem

Cayley's theorem
Every finite group of order n is isomorphic to a subgroup of S_n. The action of any element g_j of a group G on all other elements $\{g_1, \ldots, g_n\}$ can be viewed as a permutation: $\{g_j g_1, \ldots, g_j g_n\} = \{g_{p_j(1)}, \ldots, g_{p_j(n)}\}$. In this way g_j can be associated in a 1-1 way to the permutation p_j, it will follow the group multiplication (which is invertible). In this way it forms a subgroup of S_n. See Jones for further details.

Example: m roots of unity and $\mathbb{Z}_m$

$G' = (\{z_m \in \mathbb{C} \mid z_m^m = 1\}, \times\}$

$G = \mathbb{Z}_m = (\{0, 1, \dots, m-1\}, + \bmod(m))$

$\phi: G \longmapsto G'$

Homomorphic if $\forall g_1, g_2 \in G: \; \phi(g_1 \circ g_2) = \phi(g_1) \cdot \phi(g_2)$ i.e.

$\forall g_1, g_2 \in \mathbb{Z}_m: \; \phi((g_1+g_2)\bmod(m)) = \phi(g_1) \times \phi(g_2)$ where $\phi(g_1), \phi(g_2) \in G'$

As $z_m = e^{i2\pi(m/n)}$ where $m, n \in \mathbb{N}$, $\phi(g_m) = e^{i2\pi(m/n)}$

As $(g_1+g_2)/n = \alpha + (g_1+g_2)\bmod(n)/n$ where $\alpha, (g_1+g_2)\bmod(n) \in \mathbb{Z}$ we have $(g_1+g_2)\bmod(n) = g_1+g_2-\alpha n$

Then $e^{i2\pi(g_1+g_2)\bmod(n)/n} = e^{i2\pi(g_1+g_2)/n} \times e^{-i2\pi\alpha} = e^{i2\pi g_1/n} \times e^{i2\pi g_2/n}$

Then: $\phi((g_1+g_2)\bmod(n)) = \phi(g_1) \times \phi(g_2)$ if $\phi: m \longmapsto e^{i2\pi(m/n)}$ This mapping is 1-to-1

As a result, G' is isomorphic to $\mathbb{Z}_m$

Example: Euclidean Group

One can map $E(3)$ to $G' = (\{1, -1\}; \times)$ by applying $\phi: (O|\vec{\alpha}) \longmapsto \det O$

As there are two options for $\det O$, this mapping is many-to-one

As $\phi((O_2|\vec{\alpha}_2)(O_1|\vec{\alpha}_1)) = \det(O_2 O_1) = \det(O_2)\det(O_1)$ and ϕ is many-to-one this is a homomorphism

The kernel $E^+(3)$ is a subgroup of $E(3)$ constituted by rotations, translations and rotations + translations so that $\det(O) = 1$

$\hookrightarrow$ This is known as proper Euclidean group or group of "rigid motions"

Examples

1) $D_3 \cong S_3$

2) $C_m \cong \mathbb{Z}_m$

3) $SO(2) \cong U(1)$

$\hookrightarrow$ $SO(2)$: All real matrices $R^T R = 1$ and $\det R = 1$

If $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $R^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = R^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \implies d = a$ and $b = -c$

Then:

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\hookrightarrow$ $U(1) = \{e^{i\varphi} \mid \varphi \in [0, 2\pi]\}$

$\phi: SO(2) \longmapsto U(1)$, $R(\theta) \longmapsto e^{i\theta}$

As $R(\alpha)R(\beta) = R(\alpha+\beta)$ we have a homomorphism

as $R(\alpha)R(\beta) \longmapsto e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$

It is also a isomorphism

4) $\phi: E(3) \longmapsto \mathbb{Z}_2$, $(\vec{\alpha}|O) \longmapsto \det(O)$ $\forall \vec{\alpha} \in \mathbb{R}^3$, $\forall O \in O(n)$

This is a homomorphism as $(\vec{\alpha}_1|O_1)(\vec{\alpha}_2|O_2) = (\vec{\alpha}_3|O_3)$ and $\det(O_3) = \det(O_1 O_2) = \det(O_1) \cdot \det(O_2)$

Hower, as the elements of $E(3)$ are infinite but $\det(O) = \pm 1$ this is not an isomorphism

5) $\phi: G \longmapsto 1$

6) $\phi: O \longmapsto SO(3)$, $O \longmapsto \det(O)O$ as in $\mathbb{R}^m$: $\det(\lambda O) = \lambda^m \det(O)$ if m = odd

$\det(\det(O)O) = (\pm 1)^3 \det(O) = +1$

This is a homorphism but not an isomorphism as it is 2 to 1 $\ker\phi = \{\mathbb{1}_{3\times3}, -\mathbb{1}_{3\times3}\}$

7) $\phi_1: D_3 \longrightarrow C_3$ $\quad b^k c^m \longmapsto c^m$ (Remove refl.) Not Homomorphism as $\phi_1(bc)\phi_1(bc) = c^2 \neq \phi_1(bcbc) = e$

$\phi_2: D_3 \longrightarrow C_2$ $\quad b^k c^m \longmapsto b^k$ (Remove rot.) Homomorphism as $\phi_2(b^k c^m)\phi_2(b^l c^p) = b^{k+l} = \phi(b^k c^m b^l c^p) = \phi_2(b^{k+l} c^{\dots}) = b^{k+l}$

Why $\ker\phi$ must be made by complete classes of G when $\phi: G \longrightarrow G'$

$\ker\phi_1 = \{e, b\} \neq \{(e), (b)\}$

$\ker\phi_2 = \{e, c, c^2\} = \{(e), (c)\}$

Review of Linear Algebra Concepts

When dealing with representations, it is generally good practice to look at them as groups of linear transformations acting on vector spaces
This section is dedicated to reviewing some concepts fundamental for this description

Fields and Vector Spaces

Field

A field is a set F on which the binary operations of addition (+) and multiplication (·) are defined.
$\forall a,b,c,d \in F$ the field axioms are:

1) Closure: $a+b=c$ and $a\cdot b=d$
2) Associativity: $a+(b+c)=(a+b)+c$ and $a\cdot(b\cdot c)=(a\cdot b)\cdot c$
3) Commutativity: $a+b=b+a$ and $a\cdot b=b\cdot a$
4) Identities: $0\in F \mid a+0=a$ and $1\in F \mid a\cdot 1=a$
5) Inverses: $\forall a\in F, \exists (-a)\in F \mid a+(-a)=0$ and $\forall a\in F, \exists a^{-1}\in F \mid a^{-1}\cdot a=1$
6) Distributivity: $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$

(?)
Field: A Field is an ensemble of two abelian groups, one of all elements of F with addition as comb. law and a second group defined by all non-zero element with multiplication as comb. law. Multiplication and addition distribute

Vector Space

A vector space defined over a field F is a non-empty set V over which a binary operation (addition, $+: V\times V \longrightarrow V$) and a binary function (scalar multiplication, $\cdot: F\times V \longrightarrow V$) are defined. Any vectors $\vec{u},\vec{v},\vec{w},\vec{r}\in V$ satisfy the following axioms for any scalar $a,b\in F$:

1) Closure: $\vec{u}+\vec{v}=\vec{w}$ and $a\cdot\vec{v}=\vec{r}$
2) Associativity: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
3) Identity: $\vec{0}\in V \mid \vec{u}+\vec{0}=\vec{u}$ and $1\in F \mid 1\cdot\vec{v}=\vec{v}$
4) Inverse: $\forall\vec{v}\in V, \exists(-\vec{v}) \mid \vec{v}+(-\vec{v})=\vec{0}$ and $\forall a\neq 0, \exists a^{-1}\in F \mid a^{-1}(a\vec{v})=\vec{v}$
5) Commutativity: $\vec{v}+\vec{u}=\vec{u}+\vec{v}$
6) Compatibility: $(ab)\cdot\vec{v}=a(b\cdot\vec{v})$
7) Distributive
   - ↳ of vector add. wrt scalar mult.: $(a+b)\vec{v}=a\vec{v}+b\vec{v}$
   - ↳ of scalar mult. wrt vect. add.: $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$

Vector space is an abelian group under addition

Basis and Linear Independence

Linearly independent vectors

A set of vectors $\{\vec{e}_i\}$, $i=1,\dots,n$, is linearly independent if there is no non-trivial combination which yields the null vector
That is: If $\{\vec{e}_i\}$ is linearly independent $\sum_i \lambda_i\vec{e}_i=\vec{0}$ if and only if $\lambda_i=0\ \forall i$

Basis

A linearly independent set of vectors $\{\vec{e}_i\}$, $i=1,\dots,n$, forms a basis of V if they span the space i.e. any $\vec{u}\in V$ can be expressed as a vector addition of elements of the basis: $\vec{u}=\sum_i^n u_i\vec{e}_i$
If the basis has n-vectors the vector space is said to be n-dimensional while it is infinite dimensional if an infinite number of linearly independent vectors can be found

Linear Transformations

Linear Map

A map $T: V\longrightarrow V$ is linear if it satisfies the conditions the following conditions $\forall\vec{u},\vec{v}\in V$ and $\forall a\in F$:

1) Additivity: $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$
2) Scalar Mult: $T(a\vec{u})=aT(\vec{u})$

$T(a\vec{u}+b\vec{v})=aT(\vec{u})+bT(\vec{v})\quad \forall\vec{u},\vec{v}\in V$ and $a,b\in F$

Given a basis $\{\vec{e}_i\}$, T realises as a matrix D_{ij} whose elements are given by: $T\vec{e}_j=D_{ij}\vec{e}_i$
As $\vec{v}=v_i\vec{e}_i=T\vec{u}=T(u_j\vec{e}_j)=u_jD_{ij}\vec{e}_i$ we have that $v_i=D_{ij}u_j$ or $D:\ \vec{v}=D\vec{u}$

Similarity

Say that $\{\vec{e}_i\}$ and $\{\vec{\xi}_i\}$ are both basis of a vector space V.

As $\forall \vec{e}_i, \vec{\xi}_i \in V$, the two basis can be written a linear combination of the other basis vectors as follows: $\vec{e}_i = S_{ji}\vec{\xi}_j$

It follows that: $\vec{u} = u_i\vec{e}_i = u_i S_{ij}\vec{\xi}_j = u'_j\vec{\xi}_j \Longrightarrow \underline{u'_j = u_i S_{ij}}$ or $\vec{u}' = S\vec{u} \quad \forall \vec{u} \in V$

↳ relation must be invertible as both $\vec{e}_i$ and $\vec{\xi}_j$ are vectors in V i.e. $\exists S^{-1}$

Now consider a linear map $T: V \longrightarrow V$ such that $T\vec{e}_j = D_{ij}\vec{e}_i$, $T\vec{\xi}_j = D'_{ij}\vec{\xi}_i$ and $\vec{v} = T\vec{u}$

It follows that: $\vec{v}' = S\vec{v} = ST\vec{u} = SD(S^{-1}\vec{u}') = D'\vec{u}' = T\vec{u}' \Longrightarrow D' = SDS^{-1}$

A linear map manifests as different matrices (D, D') in different basis $\{\vec{e}_i\}, \{\vec{\xi}_i\}$ of the same vector space V. Nonetheless, just as two basis are related by the change of basis matrix $S: \vec{e}_i = S_{ji}\vec{\xi}_j$, so are D' and D by $D' = SDS^{-1}$. As a result, D' and D are said to be similar

Invariant Subspace

A subspace W of V is an invariant subspace for $T: V \longmapsto V$ if T maps every vector in W back into W

↳ $W \subseteq V$ is T-invariant if $\vec{v} \in W \Longrightarrow T(\vec{v}) \in W$ i.e. $TW \subseteq W$

e.g. If $T: V \longmapsto V$, the only invariant subspaces are V itself and $\{\vec{0}\}$

Scalar Product

Scalar Product on a Vector Space V

The scalar product $(\vec{u}, \vec{v})$ is defined as a binary operation/map $V \times V \longrightarrow \mathbb{C}$ which assigns each ordered pair $\vec{u}, \vec{v} \in V$ a scalar in $\mathbb{C}$.

The binary operation must satisfy the following properties:

1) Hermiticity: $(\vec{u}, \vec{v}) = (\vec{v}, \vec{u})^*$
2) Linearity: $(\vec{w}, \alpha\vec{u} + \beta\vec{v}) = \alpha(\vec{w}, \vec{u}) + \beta(\vec{w}, \vec{u})$
3) Positivity: $(\vec{u}, \vec{u}) \geqslant 0$

N.B. This is generally referred to as "dot product" in n-dimensional euclidean space. Another example is the overlap integral in wave mechanics $(\varphi, \phi) := \int \varphi^*(x)\phi(x)\, d^3x$

A vector $\vec{u} \in V$ is said to be normalized if $|\vec{u}| = (\vec{u}, \vec{u})^{1/2} = 1$

Two vectors $\vec{u}, \vec{v} \in V$ are said to be orthogonal if $(\vec{u}, \vec{v}) = \vec{0}$

Orthonormal Basis

An orthonormal basis $\{\vec{e}_i\} \in V$ satisfies $(\vec{e}_i, \vec{e}_j) = \delta_{ij}$

Given any basis $\{\vec{v}_i\}$ of V one can always construct an orthonormal basis $\{\vec{e}_i\}$ by means of the Gram-Schmidt Procedure:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{(\vec{v}, \vec{u})}{(\vec{u}, \vec{u})}\vec{u} \quad \text{and} \quad \vec{u}_k = \vec{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\vec{u}_j}(\vec{v}_k) \text{ such that } \vec{e}_k = \vec{u}_k / |\vec{u}_k|$$

Then: $(\vec{u}, \vec{v}) = (u_i\vec{e}_i, v_j\vec{e}_j) = u_i^* v_j \delta_j^i = u_i^* v_i$

Similarly, as $T\vec{e}_j = D_{ij}\vec{e}_i$ we have:

- $D_{ij} = (\vec{e}_i, T\vec{e}_j)$
- $(\vec{u}, T\vec{v}) = (u_i\vec{e}_i, v_j T\vec{e}_j) = u_i^*(\vec{e}_i, T\vec{e}_j)v_j = u_i^* D_{ij} v_j = \vec{u}^\dagger D\vec{v}$

Unitary Transformations

A linear map $T: V \longrightarrow V$ is unitary if $(T\vec{u}, T\vec{v}) = (\vec{u}, \vec{v}) \; \forall \vec{u}, \vec{v} \in V$ where $T\vec{u} = u_j T\vec{e}_j = u_j D_{ij}\vec{e}_i$

If T is invertible (i.e. $\exists T^{-1}$), it can be shown that a unitary map manifests as a unitary matrix $D^\dagger := (D^*)^T = D^{-1}$

Proof: $(T\vec{u}, T\vec{v}) = (u_j T\vec{e}_j, v_k T\vec{e}_k) = u_j^*(T\vec{e}_j, T\vec{e}_k)v_k = u_j^* D_{ij}^*(\vec{e}_i, D_{\ell k}\vec{e}_\ell)v_k = u_j^* D_{ij}^* D_{\ell k}\delta_i^\ell v_k = u^* D_{ij}^* D_{ik} v_k = u_j^* D_{ji}^\dagger D_{ik} v_k = \vec{u}^\dagger D^\dagger D\vec{v}$

$(\vec{u}, \vec{v}) = (u_j\vec{e}_j, v_k\vec{e}_k) = u_j^*\delta_k^j v_k = u_j^* v_k$

Then: $D_{ji}^\dagger D_{ik} = \delta_k^j \Longrightarrow D^\dagger D = I$ and $D^\dagger = D^{-1}$

Hermitian Transformation

A Hermitian Transformation satisfies: $(T\vec{u}, \vec{v}) = (\vec{u}, T\vec{v})$ such that $D^\dagger = D$

Representations

When applying abstract groups to physical system we need to consider the quantities on which group elements act upon.
These quantities form a carrier space for the representation of the group which manifests the action of the group on the elements of the carrier space
In most cases, the carrier space is a vector space and representations are matrix representations

Matrix Representations

Matrix Representation

A matrix representation (i.e. rep) D of dimension d of a group G is defined as a homomorphism of the group G onto the group $GL(d,K)$
If the homomorphism is an isomorphism, the rep D is said to be faithful.
Mathematically $D: G \longmapsto GL(d,K)$ s.t. $g \longmapsto D(g)$, $D(g) \in GL(d,K)$ $\forall g \in G$ and $D(g_1 \circ g_2) = D(g_1) D(g_2)$

The group $GL(d,K)$ is the general linear group of invertible (i.e. $\det \neq 0$) $d \times d$ matrices defined over a field K. Sometimes representations are defined as e elements of $GL(V)$ where V is a vector space. Nonetheless, one can establish an isomorphism between $GL(V)$ and $GL(d,K)$ once a basis for V has been determined.

Matrix Representations in different Dimensions

Consider an n dimensional Carrier Space V with orthonormal basis $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ s.t. $(\hat{e}_i, \hat{e}_j) = \hat{e}_i^\dagger \hat{e}_j = \delta_{ji}$
Each matrix representation $D(g)$ corresponds to a specific transformation $T(g): V \longmapsto V$
It follows that $T(g)\hat{e}_i = D_{ji}(g)\hat{e}_j$ and $T(g)\, b_i \hat{e}_i = b_i T(g)\hat{e}_i = b_i D_{ji}\hat{e}_j$
In addition: $(\hat{e}_k, T(g)\hat{e}_i) = \hat{e}_k^\dagger D_{ji}(g)\hat{e}_j = D_{ji}(g)\,(\hat{e}_k, \hat{e}_j) = \delta_j^k D_{ji}(g) = D_{ki}(g)$

Thanks to the relation $D_{ji}(g) = (\hat{e}_j, T(g)\hat{e}_i)$, if we know the basis of the carrier space and how $T(g)$ acts on said basis we can derive the rep D

Equivalent Representations

Consider a transformation $T(g): V \longmapsto V$ acting on the carrier space V with basis $b = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ and $b' = \{\hat{e}'_1, \hat{e}'_2, \dots, \hat{e}'_n\}$
In each basis $T(g)$ is represented by a representation s.t. $D_{ji}(g) = (\hat{e}_j, T(g)\hat{e}_i)$ and $D'_{ji}(g) = (\hat{e}'_j, T(g)\hat{e}'_i)$
As the elements of b and b' are elements of V and span V we can write each element of b' as elemnts of b by means of $S: b' \longrightarrow b$
We can thus define: $\hat{e}_i = S_{ji}\hat{e}'_j$
It follows that:

$$(\hat{e}_i, \hat{e}_j) = \delta_{ij} = S^\dagger_{ki}(\hat{e}'_k, \hat{e}'_l) S_{lj} = S^\dagger_{ki} S_{kj} \Longrightarrow S^\dagger_{ki} = S^{-1}_{ik}$$

$$D_{ji}(g) = (S_{mj}\hat{e}'_m, T(g) S_{ki}\hat{e}'_k) = S^\dagger_{mj}(\hat{e}'_m, T(g)\hat{e}'_k) S_{ki} = S^\dagger_{mj} D'_{mk}(g) S_{ki} = S^{-1}_{jm} D_{mk}(g) S_{ki}$$

$$D'_{mk}(g) = S_{mj} D_{ji}(g) S^{-1}_{ik}$$

In matrix notation: $D'(g) = S\, D(g)\, S^{-1}$

Unitary Rep

A rep $D: g \longmapsto D(g)$ is unitary if $D^\dagger_{ij}(g) D_{jk} = \delta_{jk}$ $\forall g \in G$ i.e. $D^\dagger(g) D(g) = I$ where I is the identity in $GL(d,K)$

Theorem:

If G is a finite group of order $[g]$, every rep of G is equivalent to a unitary rep
i.e. even though $D(g)$ might not be unitary, if $g \in G$ where G is a finite group, we can always find a basis in which $D'(g)$ is unitary for every g in G.

Reducibility

Reducible Representations

Definition: A rep D of a group G is reducible if it is equivalent to a rep D' for which the matrices $D'(g)$ $\forall g \in G$ is in block triagonal form

That is:

$$D'(g) = S D(g) S^{-1} = \begin{pmatrix} D_1(g) & B(g) \\ 0 & D_2(g) \end{pmatrix}$$ where D_1 and D_2 are representations themselves

One can note a couple of things:

- By choosing the appropriate basis we can greatly simplify things
- The basis transformation S should be g-independent
- The reps D_1 and D_2 might be reducible themselves, we should repeat the process until we get to irreducible representations (irreps)

Invariant Subspaces:

Let's consider the $d \times d$ matrix $D(g)$ corresponding to a transformation $T(g): V \longmapsto V$ where V is a vector space of dimension d

Let's also assume that $D(g)$ is in the block triagonal form given above, where:

- $D_1(g)$ has dimensions $m \times m$
- $B(g)$ has dimensions $m \times n$ where $n = d - m$
- $D_2(g)$ has dimensions $n \times n$

Action on the basis: $T(g)\hat{e}_i = \sum_{j=1}^{d} D(g)_{ji}\hat{e}_j = \sum_{j=1}^{m} D(g)_{ji}\hat{e}_j + \sum_{k=m+1}^{d} D(g)_{ki}\hat{e}_k =$

If $i \leq m$, $T(g)\hat{e}_i = \sum_{j=1}^{m} [D_1(g)]_{ji}\hat{e}_j$ from which

It follows that $D_1(g)$ acts on a vector space V_1 of dimension m. This is an invariant subspace of V as $D_1: V_1 \longmapsto V_1$

If $B(g) \neq \emptyset$, we cannot say the same for $D_2(g)$. However, if $B(g) = 0$, D_2 acts on the invariant subspace V_2 of dimension n

Therefore, if $D(g)$ is fully reducible i.e. $B(g) = \emptyset$

- D_1 acts on invariant subspace V_1 of dimension m spanned by $\{\hat{e}_1, \dots, \hat{e}_m\}$
- D_2 acts on invariant subspace V_2 of dimension n spanned by $\{\hat{e}_{m+1}, \dots, \hat{e}_d\}$

$\Longrightarrow$ Such that $V = V_1 \oplus V_2$

N.B. If there is an invariant subspace in V, D cannot be an irrep

Maschke's Theorem

All reducible reps of a finite group are fully reducible.

This follows from the fact that we always find an equivalent unitary rep

Examples

Example: Trivial Rep

Consider the mapping $\phi(g) = 1, \forall g \in G$ (i.e. $\phi: g \longmapsto 1 \;\; \forall g \in G$) leading to the 1 dimensional trivial rep with matrix $D(g) = (1, \times) \;\; \forall g \in G$

We can extend this to n dimensions by: $\phi: g \longmapsto I \;\forall g \in G$ where I is the identity of $GL(n, K)$

The trivial rep is commonly used for quantities that do not transform at all under the action of the group

Example: Determinant as a rep

Consider a group G with elements g of which all are defined as matrices (e.g. $O(n), U(n), \dots$). For example, consider $G = GL(d, K)$

We can then establish the mapping $\phi: G \longmapsto (\mathbb{R}\backslash\{0\}, \times)$ by $\phi(g) = \det(g)$ where $\det(g) \neq 0 \;\; \forall g \in G$

As $\det(g_1 g_2) = \det(g_1)\det(g_2)$, ϕ is a homomorphism and ϕ is rep

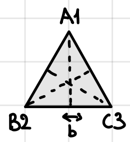
As any matrix rep D is a subgroup of $GL(d, k)$, $D' = \det(D)$ is also a rep

Example: Rep of D_3

The symmetry group of an equilateral triangle is D_3

If we define b as the reflection about the axis going throug vertex A and c as a rotation by 120° we have:

$D_3 = gp\{b, c\}$ with $b^2 = c^3 = (bc)^2 = e$



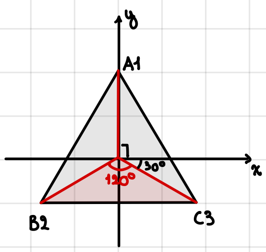
It follows that:

- $b^{-1} = b$ • $c^{-1} = c^2$ • $(bc)^{-1} = (bc)$
- $cb = bc^2$ • $c^2 b = bc$

	e	b	c	c^2	bc	bc^2
e	e	b	c	c^2	bc	bc^2
b	b	e	bc	bc^2	c	c^2
c	c	bc^2	c^2	e	b	bc
c^2	c^2	bc	e	c	bc^2	b
bc	bc	c^2	bc^2	b	e	c
bc^2	bc^2	c	b	bc	c^2	e

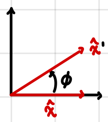
What kind of reps of D_3 are possible?

1) Trivial rep $D^{(1)}(g) = 1 \;\; \forall g \in G$ $\Longrightarrow$ Generators: $D^{(1)}(b) = D^{(1)}(c) = 1$
2) As $D_3 \cong S_3$, consider parity of permutations $\Longrightarrow$ Generators: $D^{(2)}(b) = -1 \quad D^{(2)}(c) = 1$
   ↳ $b = (23) \quad c = (123) = (12)(23)$
3) From embedding in $\mathbb{R}^2$ $\Longrightarrow$ Generators: $D^{(3)}(b) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D^{(3)}(c) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$



Rotation by ϕ on $\hat{x}$: $\hat{x} \longrightarrow \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

Rotation by ϕ on $\hat{y}$: $\hat{y} \longrightarrow -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$



Components of vectors transform according to inverse transformation so $R(\phi) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\vec{B} = (-x, -y) \quad \vec{C} = (x, -y)$

As $b = (23)$ i.e. $b: \vec{B} \longleftrightarrow \vec{C}$ we have $D^{(3)}(b) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

As $c = (123)$ i.e. $c: \vec{B} \longrightarrow \vec{C} \longrightarrow \vec{A}$ we have a rotation by 120° $\quad D^{(3)}(c) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$

We can also represent the group by embedding triangle in $\mathbb{R}^3$ in the xy plane

It follows that the 3-D rep D^V (where V means vector) are given by:

$$D^V(b) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} D^{(3)}(b) & \emptyset \\ \emptyset & D^{(2)}(b) \end{bmatrix} \qquad D^V(c) = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} D^{(3)}(c) & \emptyset \\ \emptyset & D^{(2)}(c) \end{bmatrix}$$

↳ Comes from fact that this is rotation by π about y axis

Schur's Lemmas

For every finite group, representations can be of two types only: irreducible (irreps) or completely reducible (reps)

↳ By similarity transformations (i.e. change of basis) one can block triagonalize a reducible matrix such that irreps constitute the diagonal elements.
In case of finite groups, this constitutes a block diagonalization

As irreps act on orthogonal subgroups of the subspace, reduction to diagonalized form of the representation greatly simplifies the problem.
Therefore we are often interested in finding the right basis to express reps in terms of irreps and then analising the system through those irreps and their properties.

Schur's Lemmas

Consider the following representations:

- The irrep D acting on the vector space V with dimension m s.t. D has dimensions $m \times m$ and $\forall \vec{v} \in V$, $D\vec{v} \in V$ (because irrep)
- The irrep D' acting on the vector space V' with dimension m' s.t. D' has dimension $m' \times m'$ and $\forall \vec{v}' \in V'$, $D\vec{v}' \in V'$ (because irrep)

<u>Lemma 1</u>: A matrix A with dimension $m \times m'$ ($A: V' \longmapsto V$) satisfies $D(g)A = AD'(g)$ $\forall g \in G$ if and only if A is the null matrix $\hat{O}$ or it is bijective.

$D(g)A = AD'(g) \implies A = \hat{O} \vee A$ is bijective

<u>Lemma 2</u>: If a matrix B commutes with the irrep $D(g)$ $\forall g \in G$, B is a complex multiple of the identity matrix I

Version 1: If D irrep $\wedge$ $BD = DB \implies B = \lambda I$, $\lambda \in \mathbb{C}$

Version 2: If $\exists B \neq \lambda I$ s.t. $BD = DB \implies D$ not an irrep

Version 3: If $BD = DB$, D is an irrep* iff $B = \lambda I$

* it should also include completely reducible but in finite groups all reducible matrices go to irreps

<u>Proof(s)</u>:

Define $\ker_A = \{\vec{v}' \in V' \mid A\vec{v}' = 0\}$ and $\mathrm{Im}_A = \{\vec{v} \in V \mid \vec{v} = A\vec{v}'\}$

If $D(g)A = AD'(g)$, both $\ker_A$ and Im_A are invariant subspaces for every irrep D' and D respectively

↳ $\forall \vec{v}' \in \ker_A$, $D(g)A\vec{v}' = AD'(g)\vec{v}' = 0$ i.e. $D'(g)\vec{v}' \in \ker_A$

↳ $\forall \vec{v} \in \mathrm{Im}_A$, $D(g)\vec{v} = D(g)A\vec{v}' = AD'(g)\vec{v}'$ i.e. $D(g)\vec{v} \in \mathrm{Im}_A$ as $D'(g)\vec{v}' \in V'$

As $D: V \longmapsto V$ and $D': V' \longmapsto V'$, the invariant subspaces are:

- $\{\vec{0}\}$ or V for D
- $\{\vec{0}\}$ or V' for D'

It follows that:

1) $\ker_A = \{\vec{0}\}$ and $\mathrm{Im}_A = V$
2) $\ker_A = V'$ and $\mathrm{Im}_A = \vec{0}$ i.e. $A: \vec{v}' \longmapsto \vec{0}$ $\forall \vec{v}' \in V$

Consider (1):

$$\left.\begin{array}{l} \text{As } \ker_A = \{\vec{0}\}, A \text{ is injective i.e. 1-to-1} \\ \text{As } \mathrm{Im}_A = V, A \text{ is surjective i.e. onto} \end{array}\right\} A \text{ is bijective and } m = m'$$

Consider (2):

As $\ker_A = V'$ and $\mathrm{Im}_A = \{\vec{0}\}$, A is the null matrix $\hat{O}$

Lemma 1

Now, if B is a $m \times m$ matrix that satisfies $BD(g) = D(g)B$ $\forall g \in G$ we can define $A = B - \lambda I$

If λ is an eigenvalue of B, $\det(A) = 0$

In addition: $\det(B) = \det(SS^{-1}B) = \det(SBS^{-1})$

As $DB = BD$, $AD = DA \implies A$ is $\hat{O}$ or bijective by Lemma 1

As $\det(A) = 0$, A is not invertible and thus $A = \hat{O}$ and $B = \lambda I$

Lemma 2

Schur's Lemma(s) on Abelian Groups

If G is an abelian group, $g_1 g_2 = g_2 g_1 \ \forall g_1, g_2 \in G$ s.t. $D(g_1)D(g_2) = D(g_2)D(g_1)$
It follows that if $D(g_1)$ and $D(g_2)$ are irreps, $D(g) = \lambda I$. However this is a reducible form
Thus, $D(g) = \lambda$ (i.e. Scalar) is the only option for an irrep of an abelian group

All complex irreps of an Abelian group are 1D

Remark

If $H < G$ (i.e. H proper subgroup of G) irreps of G restricted to H are not necessarily of H. In fact one might be able to find a matrix that commutes with the subset of matrices corresponding to a rep of H without commuting wit the whole set and this matrix would thus be different from $B = \lambda I$

and hence to a subset of matrices in the irrep, that there may be a matrix $B \neq \lambda I$ that commutes with this subset, but not with the whole set. Consider for example the case where H is the center $Z(G)$ of G. If G is non-Abelian and has an irrep of dimension 2 or higher, then restricting this irrep to H cannot yield an irrep of H, since the center is always Abelian and Schur's second lemma implies that all irreps of an Abelian group must be 1-dimensional (note that 1-dimensional reps are by definition irreps).

Example

Example: D_3

$D^{(1)}(c) = 1 \qquad D^{(1)}(b) = 1$

$D^{(2)}(c) = 1 \qquad D^{(2)}(b) = -1$

$D^{(3)}(c) = R(120°) \qquad D^{(3)}(b) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Set $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $D^{(3)}(c)B = BD^{(3)}(c) \Longrightarrow B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

If $D^{(3)}(b)B = BD^{(3)}(b) \Longrightarrow B = aI$

Irreps of U(1)

The group SO(2) consists of all 2x2 matrices that are orthogonal and have determinant 1. These are the rotation matrices $R(\theta)$

The group U(1) consists of all 1x1 unitary matrices. These correspond to rotations in the complex plane by $e^{i\theta}$

Therefore:

- $U(1): e^{i\theta} \ \forall \theta \in [0, 2]$
- $SO(2): R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Both are irreducible over their respective field

SO(2) is isomorphic to U(1) via the map $\phi: R(\theta) \longmapsto e^{i\theta}$ and other maps i.e. reps of U(1) are also reps of SO(2)

In addition, if we move SO(2) away from $\mathbb{R}$ and extend it to $\mathbb{C}$ we can further reduce $R(\theta)$

↳ Define $\hat{z}_1 = \hat{x} - i\hat{y}$ $\hat{z}_2 = \hat{x} + i\hat{y}$

A rotation by θ: $\hat{z}_2 \longmapsto \hat{z}_2' = (\cos\theta - i\sin\theta)\hat{x} + i(\cos\theta - i\sin\theta)\hat{y} = e^{-i\theta}\hat{x} + ie^{-i\theta}\hat{y} = e^{-i\theta}\hat{z}_2$

$\hat{z}_1 \longmapsto \hat{z}_1' = (\cos\theta + i\sin\theta)\hat{x} - i(\cos\theta + i\sin\theta)\hat{y} = e^{i\theta}\hat{x} - ie^{i\theta}\hat{y} = e^{i\theta}\hat{z}_1$

As vectors transform with inverse of basis we have $R(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ which makes $SO(2) \cong U(1)$ very clear

We thus want to find all irreps of U(1):

- Trivial rep: $D^{(0)}(\theta) = 1$
- Defining rep: $D^{(1)}(\theta) = e^{i\theta}$ "Faithful"
- Others:
  - ↳ $D^{(-1)}(\theta) = D^{(1)*}(\theta) = e^{-i\theta} \not\approx D^{(1)}(\theta)$ (Not equivalent)
  - ↳ $e^{i2\theta}, e^{i3\theta}, \dots$

Irreps of U(1) given by $D^{(m)}(\theta) = e^{im\theta}$, $m \in \mathbb{Z}$

Example

Consider the angular momentum $|l\ m\rangle$ states for $l=2$

Under 3 dimensional rotations (i.e. SO(3)) but if restrict ourselves to SO(2) e.g. rotation around z axis we have:

$$\begin{pmatrix} |2\ 2\rangle \\ |2\ 1\rangle \\ \dots \\ |2\ {-2}\rangle \end{pmatrix} \xrightarrow{R(\theta)} \begin{pmatrix} e^{i2\theta}|2\ 2\rangle \\ e^{i\theta}|2\ 1\rangle \\ \dots \\ e^{-2i\theta}|2\ {-2}\rangle \end{pmatrix}$$

i.e. $|l\ m\rangle \longrightarrow e^{im\theta}|l\ m\rangle$

↳ Irreps of U(1)

Characters

While we can find whether a rep is an irrep or not by Schur's Lemma, we also want to find all irreps up to equivalence i.e. we don't want to consider the same irreps multiple times. To do so we can use characters and character tables

Characters

Definition: Consider the d-dimensional rep D of a group G i.e. $D: G \longrightarrow GL(d,K)$. The character is the mapping $\chi^D: G \longmapsto \mathbb{C}$ such that $\chi^D(g) = Tr(D(g)) = \sum_j D(g)_{jj}$

Properties:

The Trace is cyclic: If $D = ABC$, $Tr(D) = A_{ij}B_{jk}C_{ki} = B_{jk}C_{ki}A_{ij}$

By combining this with the definition of $\chi^D(g)$ we get the following properties

1) $\chi^D(e) = d$ with $d \neq 0$

   Proof: $D(e) = I$ where I is identity in $GL(d,K)$

   $I_{ij} = \delta_{ij} \implies \chi^D(e) = Tr(D(e)) = Tr(I) = d \times \delta_{ii} = d$

2) The character is constant on the class i.e. $\chi^D(g') = \chi^D(g)$ if $g' = hgh^{-1}$

   Proof: $g' = hgh^{-1} \implies D(g') = D(h)D(g)D(h^{-1}) = D(h)D(g)D^{-1}(h)$

   $\chi^D(g') = Tr(D(g')) = Tr[D(h)D(g)D^{-1}(h)] = Tr(D(g)) = \chi^D(g)$

3) The character is independent of the basis choice: $\chi^D(g) = \chi^{D'}(g)$

   Proof: $D' \sim D$ s.t. $D' = SDS^{-1}$

   $\chi^{D'}(g) = Tr[SD(g)S^{-1}] = Tr(D(g)) = \chi^D(g)$

   N.B. One can also prove that, for finite groups, $\chi^D(g) = \chi^{D'}(g) \implies D' \sim D$ iff D and D' are irreps

Orthogonality of Characters

Orthogonality Theorem(s):

1st Theorem: Let $D^{(\mu)}$ and $D^{(\nu)}$ be two irreps of the group G of finite order $[g]$ with dimension m_μ and m_ν respectively and character $\chi^{(\mu)}$ and $\chi^{(\nu)}$.

The characters satisfy: $\boxed{\frac{1}{[g]}\sum_{g\in G}\chi^{(\mu)}(g)\chi^{(\nu)}(g)^* = \delta^{\mu\nu}}$

If $\mu \neq \nu$ i.e. $\sum_{g\in G}\chi^{(\mu)}(g)\chi^{(\nu)}(g)^* = 0$, $D^{(\mu)}$ and $D^{(\nu)}$ are inequivalent irreps

In terms of classes k_i: $\frac{1}{[g]}\sum_i k_i\chi^{(\mu)}(K_i)\chi^{(\nu)}(K_i)^* = \delta^{\mu\nu}$ where k_i is the number of elements in class K_i

Corollary: A rep $D^{(\mu)}$ of a group G is an irrep iff $\sum_{g\in G}|\chi^{(\mu)}(g)|^2 = [g]$

Theorem: If G is finite, two irreps $D^{(\mu)}$ and $D^{(\nu)}$ are equivalent iff $\chi^{(\mu)} = \chi^{(\nu)}$

↳ Proof: Remember that $\chi^{(\mu)} = \chi^{(\nu)}$ if $D^{(\mu)} \sim D^{(\nu)}$.

Now suppose $\chi^{(\mu)} = \chi^{(\nu)}$ but $D^{(\mu)} \nsim D^{(\nu)}$, which means: $\sum_{g\in G}|\chi^{(\mu)}(g)|^2 = 0$

As $|\chi^{(\mu)}(g)|^2 \geq 0 \quad \forall g \in G$ and $\chi^{(\mu)}(e) = m_\mu$ we have that $0 = m_\mu^2 + \ldots > 0$, which is impossible

2nd Theorem: Let G be a finite group of order G, with $\{K_i\}$ the set of conjugacy classes (each with number of elements m_i) and $\{D^{(\mu)}\}$ the set of irreps up to equivalence. Any two classes K_i and K_j satisfy: $\frac{1}{[g]}\sum_\mu m_i\chi^{(\mu)}(K_i)\chi^{(\mu)}(K_j)^* = \delta_{ij}$

Interpretation in terms of classes

View:

- $\{\sqrt{m_1}\chi^{(\mu)}(K_1), \ldots, \sqrt{m_k}\chi^{(\mu)}(K_k)\}$ as a vector with dimensionality k. The are r different vectors, one for each

  The 1st Orth. Th. states that the scalar product between any two of these vectors is $[g]\delta^{\mu\nu}$ i.e. they form a set of r linearly independent vectors

  As the vector space has a k dimensional basis, there can only be up to k linearly independent vectors in each set. Therefore $r \leq k$
- $\{\sqrt{m_i}\chi^{(1)}(K_i), \ldots, \sqrt{m_i}\chi^{(r)}(K_i)\}$ as a vector with dimensionality r. There are k of these vectors

  From 2nd Orth. Th. states that the scalar product between any two of these vectors is $[g]\delta^{ij}$ i.e. they are orthogonal

  As the vector space has a r dimensional basis, there can only be up to r linearly independent vectors in each set. Therefore $k \leq r$

It follows that, for a finite group, the number of inequivalent irreps is the same as the number of classes

Decomposing Reducible matrices

Scalar Product of Characters

Consider two characters of $\chi_1(g)$ and $\chi_2(g)$ of the reps D_1 and D_2 of the group G, with $g \in G$

It is convenient to define the scalar product of the two characters χ_1, χ_2 as follows: $\langle \chi_1, \chi_2 \rangle = [g]^{-1} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1})$

As for a finite group, each rep is equivalent to a unitary rep D s.t. $D^\dagger(g) D(g) = D(g^{-1}) D(g) = I$, it follows that $\chi_2(g^{-1}) = \chi_2(g)^*$ $\forall g \in G$ if G is a finite group.

We can then write the 1st Orthogonality Theorem as $\langle \chi_1^{(\mu)}, \chi_2^{(\nu)} \rangle = [g]^{-1} \sum_{g \in G} \chi^{(\mu)}(g) \chi^{(\nu)}(g)^* = \delta^{\mu\nu}$

Direct sum of matrices

If a matrix rep D of the group G is reducible, D is equivalent to D' s.t. $D'(g)$ is block triagonalised for every $g \in G$

If G is a finite group, every reducible rep D is fully reducible i.e. can be written in block diagonal form

Therefore, every reducible rep D of the finite group G can be written as follows: $D'(g) = SD(g)S = \bigoplus_\mu a_\mu D^{(\mu)}(g)$ where $\bigoplus_\mu$ is the direct sum

The direct sum is: $\;\; \llcorner$ N° of times we have $D^{(\mu)}$ along diagonal

$$\bigoplus_\mu a_\mu D^{(\mu)} = \begin{bmatrix} D_1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & D_m \end{bmatrix}$$

Reducible matrix decomposition

Theorem: If a rep D with character χ of a finite group G, is equivalent to $D' = SDS^{-1} = \bigoplus_\mu a_\mu D^{(\mu)}$, the coefficients a_μ are determined by: $a_\mu = \langle \chi^{(\mu)}, \chi \rangle$

Proof. If $D' = SDS^{-1} = \bigoplus_\mu a_\mu D^{(\mu)}$ we have $\chi(g) = \sum_\mu a_\mu \chi^{(\mu)}(g)$ and thus $a_\mu = \langle \chi^{(\mu)}, \chi \rangle$

N.B.: The coefficients a_μ are uniquely determined

Assume $\chi = \sum_i a_i \chi^{(i)} = \sum_j b_j \chi^{(j)}$ where $a_i \neq b_i$

It follows that $\sum_i (a_i - b_i) \chi^{(i)} = 0$ but as all $\chi^{(i)}$ are linearly independent we conclude $(a_i - b_i) = 0$ $\forall i$

This is a contradiction

Examples

Example: D_3

Are the following reps of D_3 irreps?

$D^{(1)}(c) = 1$ $\qquad D^{(1)}(b) = 1$

$D^{(2)}(c) = 1$ $\qquad D^{(2)}(b) = -1$

$D^{(3)}(c) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ $\qquad D^{(3)}(b) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

To be irrep, $D^{(\mu)}$ must satisfy $\sum_g |\chi^{(\mu)}(g)|^2 = \sum_i m_i |\chi^{(\mu)}(K_i)|^2 = [g]$

$[g] = 6$

$\chi^{(1)}(c) = 1 \quad \chi^{(1)}(b) = 1 \implies \sum_i m_i |\chi^{(\mu)}(K_i)|^2 = |\chi^{(1)}(e)|^2 + 2|\chi^{(1)}(c)|^2 + 3|\chi^{(1)}(b)|^2 = 1+2+3 = 6 = [g] \implies D^{(1)}$ is an irrep

$\chi^{(2)}(c) = 1 \quad \chi^{(2)}(b) = -1 \implies \sum_i m_i |\chi^{(\mu)}(K_i)|^2 = |\chi^{(2)}(e)|^2 + 2|\chi^{(2)}(c)|^2 + 3|\chi^{(2)}(b)|^2 = 1+2+3 = 6 = [g] \implies D^{(2)}$ is an irrep

$\chi^{(3)}(c) = -1 \quad \chi^{(3)}(b) = 0 \implies \sum_i m_i |\chi^{(\mu)}(K_i)|^2 = |\chi^{(3)}(e)|^2 + 2|\chi^{(3)}(c)|^2 + 3|\chi^{(3)}(b)|^2 = 4+2+0 = 6 = [g] \implies D^{(3)}$ is an irrep

What about the following rep?

$D^V(b) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad D^V(c) = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \sum_i m_i |\chi^V(K_i)|^2 = |\chi^V(e)|^2 + 2|\chi^V(c)|^2 + 3|\chi^V(b)|^2 = 9+0+1 = 10 \neq 6 = [g] \implies D^V$ is not an irrep

Character Tables

Summary of Orthogonality

Finite Group G with:

- order $[g]$
- Set of classes $\{K_1, \dots, K_k\}$
  Each class K_i has number of elements m_i
- Set of inequivalent irreps $\{D^{(\mu)}\}$ each one with dimension d_μ

1st Orthogonality Theorem: $\sum_g \chi^{(\mu)}(g)\chi^{(\nu)}(g)^* = \sum_i m_i \chi^{(\mu)}(K_i)\chi^{(\nu)}(K_j) = [g]\,\delta^{\mu\nu}$

↳ Related Theorem: $D^{(\mu)} \sim D^{(\nu)}$ iff $\chi^{(\mu)} = \chi^{(\nu)}$

↳ Corollary: $D^{(\mu)}$ is an irrep iff $\sum_g |\chi^{(\mu)}(g)|^2 = 1$

2nd Orthogonality Theorem: $\sum_\mu m_i \chi^{(\mu)}(K_i)\chi^{(\nu)}(K_j)^* = \delta_{ij}[g]$

↳ If $K_i = K_j = (e)$ we have $\sum_\mu d_\mu^2 = [g]$

↳ If $K_i = (e) \neq K_j$ we have $\sum_\mu d_\mu \chi^{(\mu)}(K_j) = 0$ which is equivalent to $\sum_\mu d_\mu \chi^{(\mu)}(g) = 0$ if $g \neq e$

N.B.

1) Abelian groups: All irreps are 1D
2) For all 1D reps, character mapping is homomorphism
3) Number of classes = Number inequivalent irreps
4) Finite Groups: $\chi = \chi' \iff D \sim D'$

Character Tables

A character table is structured as follows

	K_1	K_2	$\dots$	K_k
$D^{(1)}$	$\chi^{(1)}(K_1)$	$\chi^{(1)}(K_2)$	$\dots$	$\chi^{(1)}(K_k)$
$D^{(2)}$	$\chi^{(2)}(K_1)$	$\chi^{(2)}(K_2)$	$\dots$	$\chi^{(2)}(K_k)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$D^{(r)}$	$\chi^{(r)}(K_1)$	$\chi^{(r)}(K_2)$	$\dots$	$\chi^{(r)}(K_k)$

By 1st Orthogonality theorem: Any two rows are orthogonal
By 2nd Orthogonality theorem: Any two columns are orthogonal
We can check results by applying:

1) $\sum_\mu d_\mu^2 = 1$
2) $\sum_\mu d_\mu \chi^{(\mu)}(K_j) = 0$

Example: D_3

From previous example:

	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

However one can derive this from the orthogonality theorems as follows:

↳ 3 classes = 3 irreps

↳ There is always the trivial rep

	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	m_2	a	b
$D^{(3)}$	m_3	c	d

From $\sum_\mu |\chi^{(\mu)}(e)|^2 = \sum_\mu d_\mu^2 = [g]$ we get: $1 + m_2^2 + m_3^2 = 6$ i.e. $m_2^2 + m_3^2 = 5$ and thus $m_2 = 1$ $m_3 = 2$ or $m_2 = 2$ $m_3 = 1$

We then select $m_2 = 1$, $m_3 = 2$

Now, there are two approaches:

- Use of theorems:

From first theorem:

$$\chi^{(1)}(e)\chi^{(2)}(e) + 2\chi^{(1)}(c)\chi^{(2)}(c) + 3\chi^{(1)}(b)\chi^{(2)}(b) = m_2 + 2a + 3b = 0 \implies 1 + 2a + 3b = 0$$

$$\chi^{(1)}(e)\chi^{(3)}(e) + 2\chi^{(1)}(c)\chi^{(3)}(c) + 3\chi^{(1)}(b)\chi^{(3)}(b) = m_3 + 2c + 3d = 0 \implies 2 + 2c + 3d = 0$$

From second theorem:

$$2\left[\chi^{(1)}(e)\chi^{(1)}(c) + \chi^{(2)}(e)\chi^{(2)}(c) + \chi^{(3)}(e)\chi^{(3)}(c)\right] = 2(1 + m_2 a + m_3 c) = 0 \implies 1 + a + 2c = 0$$

$$3\left[\chi^{(1)}(e)\chi^{(1)}(b) + \chi^{(2)}(e)\chi^{(2)}(b) + \chi^{(3)}(e)\chi^{(3)}(b)\right] = 3(1 + m_2 b + m_3 d) = 0 \implies 1 + b + 2d = 0$$

It follows that $a = 1$ $b = -1$ $c = -1$ $d = 0$

- Use fact that $D^{(2)}$ is 1D and thus $\chi^{(2)}(g)$ is homomorphism

  $\chi^{(2)}(e) = 1 = \chi^{(2)}(b^2) = \chi^{(2)}(b)^2 \implies \chi^{(2)}(b) = \pm 1$

  $\chi^{(2)}(c) = 1 = \chi^{(2)}(c^3) = \chi^{(2)}(c)^3 \implies \chi^{(2)}(c) = 1, e^{i2\pi/3}, e^{i4\pi/3}$

  As $\chi^{(2)}(bc) = \chi^{(2)}(b) = \chi^{(2)}(c)\chi^{(2)}(b)$, $\chi^{(2)}(c) = 1$ and to not be equal to trivial rep $\chi^{(2)}(b) = -1$

  Using 2nd Orthogonality theorem:

  $2\left[\chi^{(1)}(e)\chi^{(1)}(c) + \chi^{(2)}(e)\chi^{(2)}(c) + \chi^{(3)}(e)\chi^{(3)}(c)\right] = 2(1 + m_2 + m_3 c) = 0 \implies c = -1$

  $3\left[\chi^{(1)}(e)\chi^{(1)}(b) + \chi^{(2)}(e)\chi^{(2)}(b) + \chi^{(3)}(e)\chi^{(3)}(b)\right] = 3(1 - m_2 + m_3 d) = 0 \implies d = 0$

Example: C_3

C_3 is a subgroup of D_3, but irreps of a group are not always the irreps of the subgroup

From properties of C_3:

- $c^3 = e \implies D^{(m)}(c)^3 = I$ and $\chi^{(m)}(c^3) =$

N.B

Use characters on finite groups, use Schur's lemma on infinite groups

Invariant Vectors

Vectors and Axial Vectors

The term "Vector" refers to vector quantities which transform according to the vector rep D^V

Therefore, vectors:

1) Rotate under rotations
2) Reflect under reflections

Note: D^V is:
- "Defining" irrep of $SO(3)$ and $O(3)$ as $SO(3) < O(3)$
- [3] rep of $SO(2)$ if one direction $\hat{x}, \hat{y}$ or $\hat{z}$ const i.e. not an irrep
- different for every group

"Axial vectors" are vector quantities which transform according to the axial vector rep D^A

Therefore, axial vectors:

1) Behave like vectors under rotation i.e. $D^A(R) = D^V(R) \quad \forall R \in SO(3)$
2) Behave opposite to vectors under reflections i.e. $D^A(P) = -D^V(P) \quad \forall P \in O(3) \setminus SO(3)$

"Scalars" are numbers which remain invariant under the action of the group i.e. transform under the trivial rep $D^{(1)}$

"Pseudoscalars" are numbers which transforms trivially but pick up a minus sign under reflections

Products of Vectors and Axial Vectors

Consider two vectors $\vec{a}$ and $\vec{b}$ with scalar elements a_i, b_i respectively

In addition consider the rotation $R \in SO(3)$ and the reflection $P \in O(3) \setminus SO(3)$ such that $R^\dagger R = P^\dagger P = \mathbb{1}$

- Inner/Scalar product: $\vec{a} \cdot \vec{b} = a_i b_j \delta^{ij} = a_i b_i$
  - ↳ Rotation $R: \vec{a}' \cdot \vec{b}' = (R_{ik} a_k)(R_{jm} b_m)\delta^{ij} = (\overset{R^\dagger_{ki}}{R_{ik}} R_{im}) a_k b_m = \delta_{km} a_k b_m$
  - ↳ Reflections $P: \vec{a}' \cdot \vec{b}' = (P_{ik} a_k)(P_{jm} b_m)\delta^{ij} = \delta_{km} a_k b_m$
- Cross product: $(\vec{a} \times \vec{b})_i = \varepsilon_{ijk} a_j b_k$
  - ↳ Reflection $P: (\vec{a}' \times \vec{b}')_i = \varepsilon_{ijk} (P_{jm} a_m)(P_{kn} b_n) = \varepsilon_{ijk} (P_{jm} P_{kn})(a_m b_n)$

    $P_{jm} P_{kn}$ is generally different from $(-\delta_{jm}\delta_{kn})$ i.e. the cross product is an axial vector

Now consider the axial vectors $\vec{c}, \vec{d}$

- Inner product: $\vec{c} \cdot \vec{d} = c_i d_j \delta^{ij}$
  - ↳ Rotation $R: \vec{c}' \cdot \vec{d}' = (R_{im} c_m)(R_{jn} d_n)\delta^{ij} = c_m d_n \delta^{mn}$
  - ↳ Reflection $P: \vec{c}' \cdot \vec{d}' = (-P_{im} c_m)(-P_{jn} d_n)\delta^{ij} = c_m d_n \delta^{mn}$
- Cross Product: $(\vec{c} \times \vec{d})_i = \varepsilon_{ijk} c_j d_k$
  - ↳ Reflection $P: (\vec{c}' \times \vec{d}')_i = \varepsilon_{ijk} (-P_{jm} c_m)(-P_{kn} d_n) = \varepsilon_{ijk} (P_{jm} P_{kn})(c_m d_n)$

    $P_{jm} P_{kn}$ is generally different from $(-\delta_{jm}\delta_{kn})$ i.e. the cross product is an axial vector

Now consider the vector $\vec{a}$ and axial vector $\vec{c}$

- Inner product: $\vec{a} \cdot \vec{c} = a_i c_j \delta^{ij}$
  - ↳ Rotation $R: \vec{a}' \cdot \vec{c}' = (R_{im} a_m)(R_{jn} c_n)\delta^{ij} = a_m c_n \delta^{mn}$
  - ↳ Reflection $P: \vec{a}' \cdot \vec{c}' = (P_{im} a_m)(-P_{jn} c_n)\delta^{ij} = -a_m c_n \delta^{mn}$
- Cross Product: $(\vec{a} \times \vec{c})_i = \varepsilon_{ijk} a_j c_k$
  - ↳ Reflection $P: (\vec{a}' \times \vec{c}')_i = \varepsilon_{ijk} (P_{jm} a_m)(-P_{kn} c_n) = -\varepsilon_{ijk} (P_{jm} P_{kn})(a_m c_n)$

    $P_{jm} P_{kn}$ is generally different from $(-\delta_{jm}\delta_{kn})$ i.e. the cross product is a vector

Therefore:

- Inner product between two (axial) vectors is a scalar e.g. $\vec{r}_1 \cdot \vec{r}_2$, $\vec{p}_1 \cdot \vec{p}_2$
- Cross product between two (axial) vectors is an axial vector e.g. $\vec{L} = \vec{r} \times \vec{p}$, $\vec{B} = \vec{\nabla} \times \vec{A}$
- Inner product between an axial vector and a vector is a pseudoscalar e.g. $\vec{L} \cdot \vec{S}$
- Cross product between an axial vector and a vector is a vector e.g. $\vec{E} \times \vec{B} = (-\vec{\nabla} V - \partial_t \vec{A}) \times (\vec{\nabla} \times \vec{A})$

Tensor and Product Representations

Products of vectors and axial vectors transform according to tensor product representations

For example, let's consider the inner product between to vectors $\vec{a}$ and $\vec{b}$ in a vector space V in $\mathbb{R}^3$

D^V: $\vec{a}\cdot\vec{b} = a_i b_j \delta^{ij} \longmapsto \vec{a}'\cdot\vec{b}' = (D^V_{in} D^V_{jm})(a_n b_m)\delta^{ij} = D^{(V\times V)}_{ij,nm}\, r_{nm}\, \delta^{ij}$

The matrix $D^{(V\times V)}$ is the result of the outer product between two D^V matrices and thus lives in $\mathbb{R}^9$ i.e. $D^{(V\times V)} = D^V \otimes D^V$ and $\mathbb{R}^9 = \mathbb{R}^3 \otimes \mathbb{R}^3$

The $\mathbb{R}^9$ vector is given by $T_{nm} = (a_1 b_1 \cdots a_2 b_2 \cdots a_3 b_3)^T$ which is a 3D tensor T_{ij} in $\mathbb{R}^3$

Note: The new $\mathbb{R}^9$ matrices are denoted by couple of indices instead of just one

Theorem: If $D^{(\mu)}$ and $D^{(\nu)}$ are two irreps of a group G with dimensions n_μ and n_ν, the matrix $D^{(\mu\times\nu)}(g) = D^{(\mu)}(g) \otimes D^{(\nu)}(g)$ (where $g \in G$) is also a rep of G of dimension $n_\mu n_\nu$. Its character is given by $\chi^{(\mu\times\nu)}(g) = \chi^{(\mu)}(g)\chi^{(\nu)}(g)$

Proof:

$$D^{(\mu\times\nu)}(g_1)\, D^{(\mu\times\nu)}(g_2) = \left(D^{(\mu)}(g_1) \otimes D^{(\nu)}(g_1)\right)\left(D^{(\mu)}(g_2) \otimes D^{(\nu)}(g_2)\right)$$

$$D^{(\mu\times\nu)}_{ij,nm}(g_1)\, D^{(\mu\times\nu)}_{nm,ab}(g_2) = \left[D^{(\mu)}_{in}(g_1) D^{(\nu)}_{jm}(g_1)\right]\left[D^{(\mu)}_{na}(g_2) D^{(\nu)}_{mb}(g_2)\right] =$$
$$= D^{(\mu)}_{ia}(g_1 \circ g_2)\, D^{(\nu)}_{jb}(g_1 \circ g_2) = D^{(\mu\times\nu)}_{ij,ab}(g_1 \circ g_2) \quad \text{Rep!}$$

$$\left.\begin{aligned} D^{(\mu)}(g) &\Longrightarrow \chi^{(\mu)}(g) = \sum_i^{n_\mu} D^{(\mu)}_{ii}(g) \\ D^{(\nu)}(g) &\Longrightarrow \chi^{(\nu)}(g) = \sum_j^{n_\nu} D^{(\nu)}_{jj}(g) \end{aligned}\right] \Longrightarrow \chi^{(\mu\times\nu)} = \sum_i^{n_\mu}\sum_j^{n_\nu} D^{(\mu\times\nu)}_{ij,ij}(g) = \sum_i^{n_\mu} D^{(\mu)}_{ii}(g) \sum_j^{n_\nu} D^{(\nu)}_{jj}(g) = \chi^{(\mu)}(g)\chi^{(\nu)}(g)$$

Clebsch - Gordan Series

Product representations are reducible i.e. by a basis transformation $\{\hat{e}\} \longmapsto \{\hat{e}'\} = S\{\hat{e}\}$ we can block diagonalize the rep

This can thus be stated as: $D^{(\mu)} \otimes D^{(\nu)} = S^{-1}\underbrace{\left(\bigoplus_\sigma a^\sigma_{\mu\nu} D^{(\sigma)}\right)S}_{n_\mu n_\nu \text{ dimensional}}$ where $a^{(\sigma)}_{\mu\nu} = \langle \chi^{(\sigma)}, \chi^{(\mu\times\nu)} \rangle$

The $a^{(\sigma)}_{\mu\nu}$ coefficients are not to be confused with the CG coefficients. The CG coefficients arise from the basis transformation

The $D^{(\mu)}$ and $D^{(\nu)}$ irreps act on orthogonal vector spaces spanned by basis $\{\varphi^{(\mu)}_n\}$ and $\{\varphi^{(\nu)}_m\}$ respectively

The rep $D^{(\mu\times\nu)}$ acts on a basis $\{\varphi^{(\mu)}_n \varphi^{(\nu)}_m\}$ such that by similarity transformation we decompose it into the separate $\{\varphi^{(\sigma)}_{\mu\nu}\}$ basis

It follows that: $\varphi^{(\sigma)\alpha}_s = \sum_{n,m} \overbrace{\begin{pmatrix} \mu\nu & \sigma\alpha \\ nm & s \end{pmatrix}}^{\text{CG Coeff}} \varphi^{(\mu)}_n \varphi^{(\nu)}_m$ where α goes from 1 to $a^{(\sigma)}_{\mu\nu}$ to label the bases of repeated irreps

Tensors

As we saw earlier, product representations act on tensors with as many indices as reps involved in the product

Let's consider the case of a product between two vectors $\vec{a}$ and $\vec{b}$ s.t. $T_{ij} = a_i b_j$ transforming through vector rep D^V

A general tensor T_{ij} can be decomposed into its symmetric and antisymmetric components $T_{ij} = \frac{1}{2}(T_{ij}+T_{ji}) + \frac{1}{2}(T_{ij}-T_{ji})$

The first term is the symmetric part and it involves 6 components for an $\mathbb{R}^3$ tensor

The second term is the antisymmetric part and it involves 3 compoonements for an $\mathbb{R}^3$ tensor

Note: Antisymmetric tensors are always traceless

We can also consider whether a tensor is traceless or not by adding the trace $Tr(T_{ij}) = T_{kk}$ to the diagonal elements

$$T_{ij} = c\delta_{ij}(T_{kk} - T_{kk}) + \frac{1}{2}(T_{ij}+T_{ji}) + \frac{1}{2}(T_{ij}-T_{ji}) = c\delta_{ij}T_{kk} + \frac{1}{2}(T_{ij}-T_{ji}) + \frac{1}{2}(T_{ij}+T_{ji} - 2c\delta_{ij}T_{kk})$$

By setting $c = 1/3$ we have the following decomposition: $T_{ij} = \frac{1}{3}\delta_{ij}T_{kk} + \frac{1}{2}(T_{ij}-T_{ji}) + \frac{1}{2}(T_{ij}+T_{ji} - (2/3)\delta_{ij}T_{kk})$

- The first term is a 1-component trace term $\Longrightarrow$ Scalar in $\mathbb{R}^3$
- The second term is the 3 component antisymmetric term $\Longrightarrow$ 3-Vector in $\mathbb{R}^3$
- The third term is the symmetric term $\Longrightarrow$ 5-Vector in $\mathbb{R}^3$

Symmetric and antisymmetric compoonements $\Longrightarrow$ They form invariant subspaces

e.g. $S_{ij} \longmapsto S'_{ij} = D^{(V\times V)}_{ij,nm} S_{nm} = D^V_{in} D^V_{jm} S_{nm} = D^V_{jm} D^V_{in} S^T_{mn}$

If S is symmetric : $S'_{ij} = D^V_{jm} D^V_{in} S_{mn} = S'_{ji}$ $\quad$ S' is symmetric

If S is antisymmetric: $S'_{ij} = D^V_{jm} D^V_{in}(-S_{mn}) = -S'_{ji}$ $\quad$ S' is antisymmetric

Thus the CG decomposition $D^{(V\times V)}$ over SO(3) is $D^{(V\times V)} \sim D_{triv} \oplus D_{3\times3} \oplus D_{5\times5}$

Thus by change of basis

$$\begin{pmatrix} T_{11} \\ \vdots \\ T_{22} \\ \vdots \\ T_{33} \end{pmatrix} \xmapsto[\text{(Dropped constants)}]{S} \begin{pmatrix} T_{kk} \\ \hline (T_{23} - T_{32}) \\ (T_{31} - T_{13}) \\ (T_{12} - T_{21}) \\ \hline (T_{ij} + T_{ji} - \frac{2}{3}\delta_{ij}T_{kk} \end{pmatrix}$$

We have thus a 1D invariant subspace (Scalar product)

+ 3D invariant subspace (Cross product)

+ 5D invariant subspace

$D^{(V\times V)} \sim D_{triv} \oplus D_{3\times3} \oplus D_{5\times5}$

Tensor transformations

A tensor $T_{ij} = \frac{1}{2}(T_{ij}+T_{ji}) + \frac{1}{2}(T_{ij}-T_{ji})$ transforms accordind to $D^{(V\times V)} = D^V \otimes D^V$

As symmetric and antisymmetric form invariant subspaces we have:

$$D^{(V\times V)} \sim D^+ \oplus D^- \quad \text{s.t.} \quad D^{(V\times V)}_{ij,mn} T_{mn} = \frac{1}{2} D^+_{ij,mn}(T_{mn}+T_{nm}) + \frac{1}{2} D^-_{ij,mn}(T_{mn}-T_{nm})$$

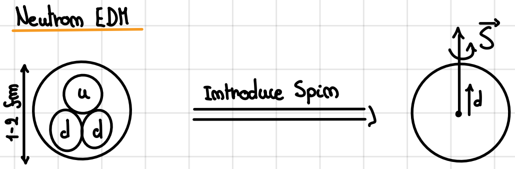
$$D^{\pm}_{ij,mn}(R) = \frac{1}{2}\left[D^V_{im}(R)D^V_{jn}(R) \pm D^V_{mj}(R)D^V_{ni}(R)\right]$$

$$\chi^{\pm}(R) = D^{\pm}_{ij,ij}(R) = \frac{1}{2}\left[D^V_{ii}(R)D^V_{jj}(R) \pm D^V_{ij}(R)D^V_{ji}(R)\right] = \frac{1}{2}\left[(\chi^V(R))^2 \pm \chi^V(R^2)\right]$$

$D^{\pm}$ can often be further decomposed but the decomposition is group specific

Examples

Neutron EDM



SO(3) or O(3)

$\hookrightarrow D^V$ irrep so no EDM

As there is a preferred direction

it is either SO(2) or O(2)

$\hookrightarrow$ If it is only a subgroup of rotation e.g. SO(2) both $\vec{S}$ and $\vec{d}$ are allowed i.e. reflection is not a symmetry

axial $\quad$ vec

$\vec{d} \propto \vec{S}$ is allowed

If it is O(2) there can't be $\vec{d}$ as there is $\vec{S}$

Conductivity Tensor

$\vec{J} = \sigma \vec{E}$ where σ is the conductivity tensor and $\vec{E}, \vec{J}$ are the electric field and current density vectors

As $\vec{J}$ and $\vec{E}$ are vectors we have:

$$\vec{J} \longmapsto \vec{J}' = D^V \vec{J} \quad \text{or} \quad j'_m = D^V_{mn} j_n = D^V_{mn}(\sigma_{nk}E_k) = \sigma'_{mn}E'_n$$

$$\vec{E} \longmapsto \vec{E}' = D^V \vec{E} \quad \text{or} \quad E'_m = D^V_{mn} E_n$$

As $\vec{J} = \sigma\vec{E}$ we have that: $\vec{J}' = \sigma'\vec{E}'$

It follows that: $j'_m = D^V_{mn}\sigma_{nk}E_k = \sigma'_{ml}E'_l = \sigma'_{ml}D^V_{lk}E_k \implies \sigma'_{ml} = D^V_{mn}\sigma_{nk}(D^V_{kl})^{-1}$ i.e. $\sigma' = D^V\sigma(D^V)^{-1}$ s.t. $\vec{J}' = D^V\vec{J} = \sigma'E'$

If D^V is real and g is a finite group, $D^V \sim U$ such that $U^\dagger U = \mathbb{1} \implies \sigma'_{ml} = D^V_{mn}(D^V_{kl})^{-1}\sigma_{nk} = (D^V_{mn})(D^V_{kl})^T\sigma_{nk} = D^{(V\times V)}_{ml,nk}\sigma_{nk}$

If Crystal has symmetry group the point group D_3 we have:

D_3	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0
D^V	3	0	-1
$D^{V\times V}$	9	0	1
D^+	6	0	2
D^-	3	0	-1

(e) is identity $\implies \chi((e)) \equiv$ dimension

(c) is class of rotation around z axis by $\theta = 120°$ $\implies \chi^V((c)) = 1+2\cos\theta = 0$

(b) is class of rotation by 180° around an axis $\implies \chi^V((b)) = 1+2\cos\theta = -1$

$\chi^{V\times V}(g) = (\chi^V(g))^2$

By decomposition: $\chi^{V\times V} \sim a_\mu D^{(\mu)} = D^{(1)} \oplus D^{(1)} \oplus D^{(2)} \oplus D^{(3)} \oplus D^{(3)} \oplus D^{(3)}$

$$a_1 = \frac{1}{6}(1\cdot1\cdot9 + 2\cdot1\cdot0 + 3\cdot1\cdot(1)) = 2$$

$$a_2 = \frac{1}{6}(1\cdot1\cdot9 + 2\cdot1\cdot0 + 3\cdot(-1)\cdot(1)) = 1$$

$$a_3 = \frac{1}{6}(1\cdot2\cdot9 + 2\cdot(-1)\cdot0 + 3\cdot(0)\cdot(1)) = 3$$

The trivial rep appears twice i.e. there exist the possibility of two invariant tensors

If σ is invariant $\sigma' = \sigma$ s.t. $D^V\sigma = \sigma D^V$

By explicit solution we find that $\sigma = a\mathbb{1} + (a-i)\begin{pmatrix}0 & & \emptyset\\ & 0 & \\ \emptyset & & 1\end{pmatrix}$ and both terms are symmetric and separately invariant

Electric and Magnetic Dipole moments

Electric dipole is an invariant vector $\implies D^V \sim D_{triv} \oplus \ldots$ if it exists

Magnetic dipole is an invariant axial vector $\implies D^A \sim D_{triv} \oplus \ldots$ if it exists

<u>Note</u>: An inv. vec. and an ax. vec cannot exist at the same time if group contains reflections

Continuous Groups

In physics, many continuous groups are important. Of largest importance are "Lie Groups".

Definition: A Lie group is a continuous group whose elements are determined by a set of parameters
The number of parameters is known as the dimension of the group
In order to consider all elements of the Lie group, we can infinitesimal generators of the group which form the "Lie Algebra"

Lie Group U(1)

Unit circle

$U(1)$ is the Lie group corresponding to all unitary 1×1 matrices U (i.e. $U^\dagger U = 1$)
It is the abelian group of complex phases $z = e^{i\alpha} \Longrightarrow U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$ and it is thus a multiplicative subgroup of $\mathbb{C}\setminus\{0\}$

Lie Group SO(2)

$SO(2)$ is the group of 2×2 orthogonal matrices O with determinant one (i.e. $O^T O = 1$ with $\det(O) = 1$)
The group is abelian and is often viewed as the group of proper rotations in 2 dimensions with rep $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
As $R(\theta)$ is an isomorphism, the rep is faithful and corresponds to $SO(2)$ itself

As $R(\theta)$ is abelian the irreps are all 1D. The irreps turn out to be the irreps of $U(1)$ i.e. $D^{(m)}(\phi) = e^{im\phi}$ with $m \in \mathbb{Z}$ s.t. $R(\theta) \sim D^{(m)}(\phi) \oplus D^{(-m)}(\phi)$
As $SO(2)$ is a compact group with $\theta \in [0, 2\pi)$ and therefore we can extend orthogonality as:

$$\langle \chi^{(m)}, \chi^{(m')} \rangle = \int_0^{2\pi} \frac{d\phi}{2\pi}\, \chi^{(m)}(\phi)\, \chi^{(m')*}(\phi) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{i(m-m')\phi} = \delta_{mm'}$$

Thus coefficient in CG Coefficients can be written as: $D^{(m)} \otimes D^{(m')} = D^{(m\times m')} \sim a^{(m)}_{mm'} D^{(m)}$ where $a^{(m)}_{mm'} = \langle \chi^{(m)}, \chi^{(m\times m')} \rangle = \delta_{m, m+m'}$

Computation of the generator

To compute the generator we Taylor expand the defining representation as: $R(\theta) = \mathbb{1} + \theta\left[\frac{dR}{d\theta}\right]_{\theta=0} + \theta^2\left[\frac{d^2R}{d\theta^2}\right]_{\theta=0} + \mathcal{O}(\theta^3)$
By differentiation of $R(\theta)$ w.r.t. θ we have $(dR/d\theta)|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\mathcal{J}_z$ in defining rep

As $R(\theta) \sim e^{i\theta} \oplus e^{-i\theta}$, properties of the reduced rep will hold for the defining representation as well
Due to cyclic properties of these derivatives we have: $(d^mR/d\theta^m)_{\theta=0} = \begin{pmatrix} (i)^m & 0 \\ 0 & (-i)^m \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^m = (dR/d\theta)^m$

In addition, from the properties of R we can see:

- Orthogonality implies $\mathcal{J}_z$ is Hermitian : $R^\dagger(\theta)R(\theta) = \mathbb{1} = (\mathbb{1} + i\theta\mathcal{J}_z^\dagger + \mathcal{O}(\theta^2))(\mathbb{1} - i\theta\mathcal{J}_z + \mathcal{O}(\theta^2)) =$
$\approx \mathbb{1} + i\theta(\mathcal{J}_z^\dagger - \mathcal{J}_z)$ s.t. $\mathcal{J}_z^\dagger = \mathcal{J}_z$

We can thus write: $R(\theta) = \mathbb{1} - i\theta\mathcal{J}_z + (-i)^2\theta^2\mathcal{J}_z^2 + \mathcal{O}(\theta^3)$ whe $\mathcal{J}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the generator of rotations around $\hat{z}$-axis in the defining rep
Clearly this is the exponential expansion of the generator. $R(\theta) = \exp(-i\theta\mathcal{J}_z)$

Lie Group SO(3)

$SO(3)$ is the group of 3×3 orthogonal matrices with determinant one i.e. O s.t. O^tO and $\det(O) = 1$
This correspond two the group of rotations $R(\theta)$ along an axis $\hat{n}$. The dimension of the group is thus 3 as the angle of rotation + 2 other angles must be specified for the direction of $\hat{n}$

A simple extension of $SO(2)$ leads to the subgroup of rotations about $\hat{z}$-axis of $SO(3)$
By similar approach as in $SO(2)$ we have $R(\theta, \hat{z}) = \exp(-i\theta\mathcal{J}_z)$
Similarly: $R(\theta, \hat{x}) = \exp(-i\theta\mathcal{J}_x)$ and $R(\theta, \hat{y}) = \exp(-i\theta\mathcal{J}_y)$

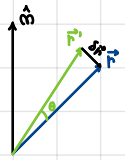
$$R(\theta, \hat{z}) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad \mathcal{J}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Denote $\hat{x} = \hat{x}_1$, $\hat{y} = \hat{x}_2$ and $\hat{z} = \hat{x}_3$

Therefore: $(\mathcal{J}_k)_{ij} = -i\epsilon_{ijk}$

Rotation about $\hat{n}$



$R(\theta,\hat{n}): \vec{r} \longmapsto \vec{r}' = R(\theta,\hat{n})\vec{r}$ s.t. $\delta\vec{r} = \vec{r}' - \vec{r}$

If θ small: $\delta\vec{r} \approx \theta(\hat{n}\times\vec{r})$

It follows that: $\vec{r}' = \vec{r} + \theta(\hat{n}\times\vec{r}) = \vec{r} - \theta(\vec{r}\times\hat{n})$

$$r_i' = r_i + \theta(\varepsilon_{ikj} n_k r_j) = r_i - \theta(\varepsilon_{ijk} r_j n_k) = [\delta_{ij} - i\theta(-i\varepsilon_{ijk} n_k)] r_j$$

$$r_i' = R_{ij} r_j \implies R_{ij}(\theta,\hat{n}) = \delta_{ij} - i\theta n_k (J_k)_{ij}$$

We can thus write: $R(\theta,\hat{n}) = \exp(-i\theta\,\hat{n}\cdot\vec{J})$ where $\vec{J} = J_k \hat{x}_k$

Commutators $[J_i, J_j] = i J_k$

Conjugacy Classes

Consider the rotation $R(\theta,\hat{n}_R)$ and any other rotation $S(\phi,\hat{n}_S)$ in SO(3)

Then, the conjugate to R is given by $R'(\theta',\hat{n}_R') = S(\phi,\hat{n}_S)\,R(\theta,\hat{n}_R)\,S^{-1}(\phi,\hat{n}_S) =$

$$= \exp\left[(-i\phi\hat{n}_S\cdot\vec{J}) + (-i\theta\hat{n}_R\cdot\vec{J}) + (i\phi\hat{n}_S\cdot\vec{J})\right] =$$

$$= \exp\left[-i(\phi+\theta)(\hat{n}_S+\hat{n}_R)\cdot\vec{J} + (i\phi\hat{n}_S\cdot\vec{J})\right]$$

$=$

Irreps of SO(3) and SU(2)

Irreducible matrices that satisfy commutation relations are given by

- J_3
- $J_\pm = J_1 \pm J_2$

These all commute with J^2 which has eigenvalue $j(j+1)$ and j_z has $2j+1$ eigenvalues $m = -j, -j+1, \dots, +j$

We thus label irreps as $D^{(j)}$ and their invariant spaces are $2j+1$ dimensional

If j is an integer, these are irreps of SO(3) and SU(2)

If j is a half-integer, these are irreps of SU(2) only

Transformation of wavefunctions

Consider a transformation $T(g)$ such that:

$T(g)$: $\psi \longmapsto \psi'$ where the wavefunction is the basis vector

$T(g)$: $\vec{r} \longmapsto \vec{r}'$ where $\vec{r}$ is the position vector

It follows that $\psi'(\vec{r}') = \psi(\vec{r})$ and $\psi'(\vec{r}) = \psi(T^{-1}(g)\vec{r}) = U(g)\psi(\vec{r})$ where $U: G \longmapsto G'$, G' being the group of operators

Then: $U(g)\psi'(T(g)\vec{r}) = \psi(\vec{r}) \quad \forall g \in G$

$U(g_1)U(g_2)\psi = U(g_1 \circ g_2)\psi \quad \forall g_1, g_2 \in G$ Homomorphism

For probability to be conserved we have: $U^\dagger(g)U(g) = \mathbb{1}$ i.e. U must be unitary operator rep

Basis and Reps

Consider the d-dimensional set of wavefunctions created by the action of $U(g)\ \forall g \in G$ i.e. $\{\psi_g \mid \psi_g = U(g)\psi \ \ \forall g \in G\}$

An orthonormal basis $\{\phi_m\}$ of this set can be constructed by Graham-Schmidt Orthogonalization

It follows that:

$$\psi_g = \sum_{m=1}^{d} c_m \phi_m \implies U(g)\phi_k(\vec{r}) = \sum_{m=1}^{d} \phi_m(\vec{r}) D(g)_{mk} \quad \text{with } k = 1, \dots, d$$

$D(g)$ is a d-dimensional rep of G' over the space spanned by orthonormal basis $\{\phi_m\}$ such that $D(g)_{mn} = \langle \phi_m | U(g) | \phi_n \rangle$

If there are invariant subspaces, D is reducible

Irreps of SO(3) and SU(2) in context of wavefunctions

As we saw earlier, each $(2j+1)$-dimensional invariant subspaces is acted upon by irreps $D^{(j)}$

In the context of quantum mechanics, these invariant subspaces are spanned by the eigenvector basis $\{|j\,m\rangle\}$ in which $m = -j, -j+1, \dots, j-1, j$

The irreps for each of these subspaces are given by $D^{(j)}(R)_{m'm} = \langle j\,m' | U(R) | j\,m \rangle$ and it is an irrep of SO(3) and SU(2) (SU(2) only for half-integer spin)

Operator $U(R)$:

$\vec{r}' \approx \vec{r} + \delta\vec{r}$ where $\delta\vec{r} = \theta(\hat{n} \times \vec{r})$ if θ small

$U(R)\psi(\vec{r}) = \psi(R^{-1}r) = \psi(\vec{r} - \theta(\hat{n} \times \vec{r})) \approx \psi(\vec{r}) - \theta\hat{n} \cdot (\vec{r} \times \underset{i/\hbar\, p_i}{\vec{\nabla}})\psi(\vec{r})$

By exponentiation and generalisation: $U(R) = \exp(-\frac{i\theta}{\hbar}\hat{n} \cdot \vec{J})$ is the angular momentum operator

It reduces to generator of SO(3) if $\vec{J} = \vec{L}$ or j integer

The irreducible matrices that satisfy commutation relations with each other and $\underset{\text{eigenvalue } \hbar^2 j(j+1)}{J^2}$ are:

- J_z with eigenvalue $\langle j\,m' | J_z | j\,m \rangle = \hbar m\, \delta_{m'm}$
- $J_\pm$ with eigenvalue $\langle j\,m' | J_\pm | j\,m \rangle = \hbar[(j \mp m)(j \pm m + 1)]^{1/2} \delta_{m', m\pm 1}$

Examples: Hydrogen Atom Wavefunctions

$\psi_{nlm}(\vec{r}) = \langle \vec{r} | n\,l\,m \rangle = R_{nl}(\vec{r}) Y_{lm}(\theta, \phi)$

Under rotation $m \longmapsto m'$ i.e. $U(R)\psi_{nlm}(\vec{r}) = \sum_{m'} \psi_{nlm'} D^{(l)}_{m'm}(R)$ or $U(R)|l\,m\rangle = \sum_{m'} |l\,m'\rangle D_{m'm}(R)$

- Case 1: $l = 0$

  $\hookrightarrow$ Basis is $\{|0\,0\rangle\}$ i.e. 1D

  As m can only be zero if $l = 0$, $m = m' = 0$

  State is thus invariant i.e. $D^{(0)}(R) = D_{triv}$

- Case 2: $l = 1$

  $\hookrightarrow$ Basis is $\{|1\,1\rangle, |1\,0\rangle, |1\,-1\rangle\}$ i.e. 3D $\implies$ Transforms as D^V of SO(3)?

  Consider spherical basis $\{\hat{x}, \hat{y}, \hat{z}\} \longmapsto \{-(\hat{x}+i\hat{y})\sqrt{2}, \hat{z}, (\hat{x}-i\hat{y})\sqrt{2}\}$

  In this new basis the unit vectors $\{\hat{x}, \hat{y}, \hat{z}\} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$ can be written as $\frac{1}{\sqrt{2}}(-\sin\theta e^{-i\phi}, \sqrt{2}\cos\theta, \sin\theta e^{-i\phi})$

  This is equivalent to $\sqrt{\frac{4\pi}{3}}(Y_{11}, Y_{10}, Y_{1-1}) = \sqrt{\frac{4\pi}{3}}(|1\,1\rangle, |1\,0\rangle, |1\,-1\rangle)$

  As $(\hat{x}, \hat{y}, \hat{z})$ transform according to D^V so do $(|1\,1\rangle, |1\,0\rangle, |1\,-1\rangle)$

Therefore, for a rotation around z

$$D^V = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \sim D^{(1)} = \begin{pmatrix} e^{i\theta} & & \emptyset \\ & 1 & \\ \emptyset & & e^{-i\theta} \end{pmatrix} = e^{i\theta \begin{pmatrix} 1 & & \emptyset \\ & 0 & \\ \emptyset & & -1 \end{pmatrix}} = e^{-i(\theta/\hbar) L_z}$$ where L_z is generator in spherical basis

- Case 3: $\ell = 2$

↳ Basis is 5D so $D^{(2)}$ is a 5×5 irrep

The elements of $D^{(2)}$ in the basis $\{|2\,\ell\rangle, \dots, |2\,-\ell\rangle\}$ are given by:

$$D^{(2)}_{m'm} = \langle 2\,m'|\,U(R_z)\,|2\,m\rangle = \langle 2\,m'|\exp(-i\tfrac{\theta}{\hbar}L_z)|2\,m\rangle = \langle 2\,m'|\exp(-im\theta)|2\,m\rangle = e^{-im\theta}\delta_{m'm}$$ as $L_z|2\,m\rangle = \hbar m|2\,m\rangle$

It follows that:

$$D^{(2)} = e^{-i\theta \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 0 & & \\ & & & -1 & \\ & & & & -2 \end{pmatrix}}$$

We can find similar matrices for L_x, L_y by using L_+ an L_-

Addition of Angular Momentum and Clebsch-Gordan Series

In $2j+1$ rep, $J_3 = \mathrm{diag}(j, j-1, \dots, -j+1, j)$ such that $\langle j\,m'|J_3|j\,m\rangle = \hbar m\,\delta_{mm'}$

It follows that, for a rotation around the z axis (see Example above) the representation $D^{(j)}(R_z)$ has elements $D^{(j)}_{m'm}(R_z) = e^{-im\theta}\delta_{m'm}$

Its character is thus: $\chi^{(j)}(R_z) = e^{-ij\theta} + e^{-i(j-1)\theta} + \dots + e^{ij\theta} = \sin(j+\tfrac{1}{2})\theta / \sin(\tfrac{1}{2}\theta)$

As all rotations have same character: $\chi^{(j)}(\theta) = \sin[(j+\tfrac{1}{2})\theta]/\sin(\tfrac{1}{2}\theta)$ for a rotation by θ around any axis

We often deal with states of the kind $|j_1\,m_1\rangle|j_2, m_2\rangle$ which transform as $D^{(j_1\times j_2)} = D^{(j_1)} \otimes D^{(j_2)}$

What is the structure of $D^{(j_1)} \otimes D^{(j_2)}$?

We know that $\chi^{(j_1\times j_2)} = \chi^{(j_1)}\chi^{(j_2)} = \sum_{j_1-j_2}^{j_1+j_2} \chi^{(j)} \Longrightarrow D^{(j_1\times j_2)} = D^{(j_1)} \otimes D^{(j_2)} = \sum_{|j_1-j_2|}^{j_1+j_2} \oplus D^{(j)}$

It follows that: $$|j_1, m_1; j_2, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} \overbrace{\langle j_1, m_1; j_2, m_2|j, m\rangle}^{\text{CG coefficient}} |j, m\rangle$$

Example:

- $S_1 = S_2 = 1/2$

$V^{(S)}$ has a $(2S+1)$ dimensional basis and for $s = 1/2$ the basis is $\{|\uparrow\rangle, |\downarrow\rangle\}$

Then: $V^{(1/2)} \otimes V^{(1/2)}$ has basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\} \sim \{|\uparrow\downarrow\rangle, \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), |\downarrow\downarrow\rangle, \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\}$

The first 3 states are symmetric while the 4^{th} state is antisymmetric

Symm. and Anti-Symm part do not mix we have two invariant subspaces i.e. $V^{(1/2)} \otimes V^{(1/2)} = V^{(1)} \otimes V^{(0)} \Longrightarrow D^{(s_1)} \otimes D^{(s_2)} \sim \sum_{j=|j_1-j_2|}^{j_1+j_2} \oplus D^{(j)}$